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We prove $X \to F$ is a homotopy equivalence with the same argument used in (5.6) to show P_k implies H_k . Since F is also the fibre of $X_N^+ \to [B\pi_1(X)]_N^+$ we have proved the theorem.

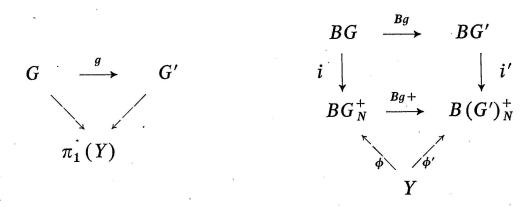
(5.8) Remark. Using (5.1), we see that for an acyclic map $f: X \to Y$ which is k-simple for all $k \ge 2$, the homotopy groups $\pi_*(Y)$ can be computed in terms of $\pi_*(X)$ and $\pi_*(B\pi_1(X)_N^+) \cong \pi_*(BN)^+$ for $i \ge 2$. Some computations of $\pi_*(BN^+)$ for a certain perfect group N can be found for instance in [H, Chapter 7].

§ 6. ACYCLIC MAPS INTO A GIVEN SPACE

In this section we study acyclic maps $f: X \to Y$ into a fixed space Y. Two such map $f: X \to Y$ and $f': X' \to Y$ are called equivalent provided there is a homotopy equivalence $h: X \to X'$ with $f \simeq f'h$. Let AC(Y) denote the class of equivalence classes of acyclic $f: X \to Y$ over Y where X and Y are CW-spaces.

- (6.1) Definition. An extension data over a space Y is a triple (Φ, i, Φ) where
- (a) Φ is an extension $1 \to N \to G \to \pi_1(Y) \to 1$ with N perfect,
- (b) $i: BG \to BG_N^+$ is an acyclic map with $\ker(\pi_1(i)) = N$ (whose equivalence class is well defined by (3.5)), and
- (c) $\phi: Y \to BG_N^+$ is a 2-connected map.

Two triples of extension data (Φ, i, ϕ) and (Φ', i', ϕ') are called equivalent provided there exists an isomorphism $g: G \to G'$ making the following diagrams commutative (up to homotopy for the second one).



where N' = g(N) and Bg^+ is the unique homotopy equivalence determined by g with (3.1).

We denote by ED(Y) the class of equivalence classes of extension data.

- (6.2) DEFINITION. The data map ρ is the function $\rho: AC(Y) \to ED(Y)$ which assigns to an acyclic map $f: X \to Y$ the class $\rho(f) = (\Phi, i, \phi)$ of extension data defined as follows:
- (a) Φ is the extension $1 \to \ker \pi_1(f) \to \pi_1(X) \to \pi_1(Y) \to 1$.
- (b) (c) With the well defined $j: X \to BG$ for $G = \pi_1(X)$ we form the cocartesian diagram

$$\begin{array}{ccc}
X & \xrightarrow{j} & BG \\
f \downarrow & & \downarrow i \\
Y & \xrightarrow{\phi} & Y \cup BG \\
\end{array}$$

Since f is acyclic, i is acyclic, and since $\pi_1(j)$ is an isomorphism, ker $(\pi_1(i))$ = N. Thus $Y \cup_X BG$ is BG_N^+ up to equivalence.

Now we have to check that the map $\phi: Y \to Y \cup_X BG = BG_N^+$ is 2-connected. Since $\pi_1(j)$ is an isomorphism, $\pi_1(\phi)$ is also an isomorphism. The fact that $\pi_2(\phi)$ is surjective comes from the diagram.

$$\pi_{2}(Y) \stackrel{\sim}{\longleftarrow} \pi_{2}(\tilde{Y}) \stackrel{\sim}{\longrightarrow} H_{2}(\tilde{Y}) \stackrel{\sim}{\longleftarrow} H_{2}(\tilde{X}_{N})$$

$$\pi_{2}(\phi) \downarrow \qquad \qquad \downarrow \pi_{2}(\tilde{\phi}) \qquad \qquad \downarrow \qquad \swarrow$$

$$\pi_{2}(BG_{N}^{+}) \stackrel{\sim}{\longleftarrow} \pi_{2}(BN^{+}) \stackrel{\sim}{\longrightarrow} H_{2}(N)$$

The surjectivity on the right is a classical result of Hopf which follows easily from the Serre spectral sequence of the fibration $\tilde{X} \to \tilde{X}_N \to BN$.

Now using (2.5) a simple argument, left to the reader, shows that $\rho: AC(Y) \to ED(Y)$ is well defined.

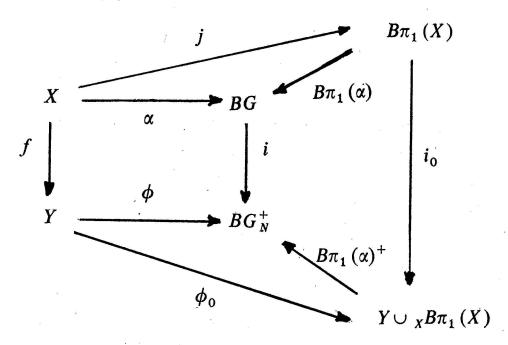
(6.3) THEOREM. Let Y be a CW-space. The map $\rho: AC(Y) \to ED(Y)$ surjective and its restriction to the subclass $AC_S(Y)$ of AC(Y) of $f: X \to Y$ which are k-simple for all $k \ge 2$ is a bijection.

Proof. To show ρ is surjective, consider extension data (Φ, i, ϕ) and form the cartesian square

$$\begin{array}{cccc} X &=& Y \times {}_T BG & \stackrel{\alpha}{\longrightarrow} & BG \\ & f & & \downarrow i \\ & Y & \stackrel{\phi}{\longrightarrow} & BG_N^+ = T \end{array}$$

Now f is acyclic by (2.2), and since its fiber is the same as i, we deduce by (5.2) that f is k-simple for all $k \ge 2$.

Next, let $\rho(f) = (\Phi_0, i_0, \phi_0)$ and we show this extension data is equivalent to (Φ, i, ϕ) . Using the homotopy exact sequences for $X \to Y$ and $BG \to BG_N^+$ and the fact that ϕ is 2-connected, we deduce from the five lemma that $\pi_1(\alpha) : \pi_1(X) \to G$ is an isomorphism. The following diagram shows that (Φ_0, i_0, ϕ_0) is equivalent to (Φ, i, ϕ) and ρ is surjective.



Now, if $f: X \to Y$ is an acyclic map which is k-simple for all $k \ge 2$ and with $\rho(f) = (\Phi, i, \phi)$, then we form the following commutative diagram.

$$\begin{array}{cccccc}
X & \xrightarrow{d} & Y \times_{T} BG & \longrightarrow & BG \\
\downarrow & & \downarrow & i & (G = \pi_{1}(X)) \\
Y & \xrightarrow{\phi} & BG_{N}^{+} & & & & & & & \\
\end{array}$$

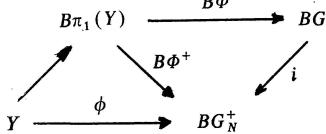
As we have seen in the proof the surjectivity of ρ , the map f_0 is acyclic and k-simple for $k \geq 2$. The map d induces an isomorphism on the fundamental groups and on homology with $\mathbb{Z} \pi_1(Y)$ twisted coefficients. By (5.3), the map d is a homotopy equivalence. This proves that the acyclic map f is equivalent to the induced map f_0 . Thus ρ restricted to $AC_S(U) \to ED(Y)$ is a bijection.

(6.4) Remark. This theorem leaves open the question of the fibres of the function.

$$\rho: AC(Y) \to ED(Y)$$
.

In the next theorem we factor an acyclic map by ones having simplicity properties.

(6.5) Remark. In theorem (6.3), if one fixes an extension $\Phi: 1 \to N \to G \to \pi_1(Y) \to 1$, then the same proof permits us to classify acyclic maps $f: X \to Y$ which are k-simple for k > 2 together with an identification $d: \pi_1(X) \to G$ such that $\Phi d = \pi_1(f)$. The objects of ED(Y) have to be replaced by couples (i, ϕ) where $i: BG \to BG_N^+$ is as above and $\phi: Y \to BG_N^+$ is 2-connected with the following diagram commuting up to homotopy.



This is what is done implicitely in [H, Sections 2 and 4]. Observe that we are dealing here with classes which are sets.

- (6.6) Lemma. Let X be a CW-space and N a perfect normal subgroup of $\pi_1(X)$. Let $X \to P_n X$ denote the nth stage of the Postnikov decomposition of X. Then for all $n \ge 1$ we have that
- (1) $\pi_j(X_N^+) \to \pi_j((P_n X)_N^+)$ is an isomorphism for $j \leq n$ and an epimorphism for j = n + 1, and
- (2) $\pi_j(A\widetilde{X_N}) \to \pi_j(A(P_n\widetilde{X_N}))$ is an isomorphism for $j \leq n$ and an epimorphism for j = n + 1.

Proof. Consider the following homotopy commutative diagram of fibre sequences

$$T \longrightarrow A\widetilde{X}_{N} \longrightarrow A(P_{n}\widetilde{X})$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$F \longrightarrow \widetilde{X}_{N} \longrightarrow P_{n}\widetilde{X}_{N}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$G \longrightarrow (\widetilde{X}_{N})^{+} \longrightarrow (P_{n}\widetilde{X}_{N})^{+}.$$

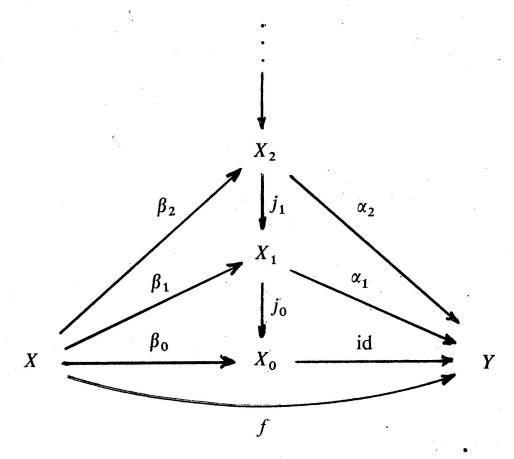
Clearly $\pi_i(F) = 0$ for $i \leq n + 1$. The spaces X_N and $P_n X_N$ have the same (n+1)-skeleton and the same can be assumed for X_N^+ and $(P_n X_N^-)^+$. Hence $\pi_i(G) = 0$ for $i \leq n + 1$. Now (1) follows because G is the fibre of X_N^+ $\to (P_n X)^+$.

By comparing Serre spectral sequences, we obtain the surjectivity of

$$H_0(N, H_{n+1}(F)) \to H_0(N, H_{n+1}(G)) = H_{n+1}(G) = \pi_{n+1}(G).$$

Thus $\pi_j(T) = 0$ for $j \le n$ and (2) follows.

(6.7) THEOREM. Let $f: X \to Y$ be a map between CW-spaces. Then there is a factorization



such that β_i is i-connected and α_i is an acyclic map which is k-simple for k > i.

Such a decomposition is unique up to a homotopy equivalence.

Proof. The ith stage X_i is defined by the cartesian diagram

$$Y \times_{T} P_{i}(X) \longrightarrow P_{i}X$$

$$\downarrow \qquad \qquad \downarrow$$

$$Y \longrightarrow (P_{i}X)_{N}^{+} = T$$

where $N = \ker (\pi_1(X) \to \pi_1(Y))$. By (6.6) the map β_i is *i*-connected since the fiber of the two vertical arrows is $A(P_n\tilde{X})_N$. Now by (5.4) we see that α_i is simple for k > i.

For two decompositions (X_i') and (X_i'') of $f: X \to Y$ satisfying the above conditions, we have $P_i X_i' = P_i X_i''$ and both X_i' and X_i'' map into X_i , constructed above, such that the resulting diagrams are homotopy commutative. The connectivity of the β_i and (5.1) shows that these maps are all homotopy equivalences. This proves the theorem.

(6.8) Remarks. This theorem (6.7) coincides with the Dror results for Y a point [D1, Theorem 1.3] and $Y = S^n$ [D2]. An interesting problem is to describe the ith stage X_i in terms of invariants of X_{i-1} as in [D1] and [D2]. (See the footnote in the introduction.)

APPENDIX — SIMPLICITY PROPERTIES OF FIBERS

In the proof of (5.4) we used the fact that for a fibration $F \to E \xrightarrow{f} B$ the action of $\pi_1(F)$ on Im $(\partial : \pi_{k+1}(B) \to \pi_k(F))$ is trivial. This assertion does not seem to be in the literature so we include a proof here.

We extend the mapping sequence of the fibration f to $\Omega B \to F \to E \xrightarrow{f} B$ and study F as the total space of a principal fibration with fibre the H-space ΩB . If G is an H-space, then $\pi_1(G)$ acts trivally on $\pi_*(G)$ because the covering transformations $G \to G$ on the universal covering G of G are homotopic to the identity. This is proved by lifting a loop to a path in G and using the G-space structure on G to deform the identity along this path to the covering transformation defined by the homotopy class of the loop. Recall that a principal fibration is induced from $G \to E_G \to B_G$ up to fibre homotopy equivalence.

- (A.1) PROPOSITION. Let $G \to X \xrightarrow{\pi} Y$ be a principal fibration with fibre G acting on X. Then we have:
- (a) im $(\pi_1(G) \to \pi_1(X))$ acts trivially on $\pi_*(X)$, and
- (b) $\pi_1(X)$ acts trivially on $\operatorname{im}(\pi_*(G) \to \pi_*(X))$.

Proof. For (a) we have the following commutative diagram induced by a covering transformation $T: G \to G$.