

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 25 (1979)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ACYCLIC MAPS
Autor: Hausmann, Jean-Claude / Husemoller, Dale
Kapitel: §5. k-SIMPLE ACYCLIC MAPS
DOI: <https://doi.org/10.5169/seals-50372>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 07.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

(4.4) *Remark.* The group $\tilde{N} = \pi_1(A\tilde{X}_N)$ is a central extension of N (see the appendix) and, as $A\tilde{X}_N$ is acyclic, satisfies $H_1(\tilde{N}) = H_2(\tilde{N}) = 0$. Therefore \tilde{N} is the universal central extension of N (see [K2]), namely one has the exact sequence $0 \rightarrow H_2(N) \rightarrow \tilde{N} \rightarrow N \rightarrow 1$. Therefore, if $f: X \rightarrow X'$ is a map such that $\pi_1(f)$ sends the perfect normal subgroup N of $\pi_1(X)$ isomorphically onto a normal subgroup N' of $\pi_1(X')$, then the induced map $Af: A\tilde{X}_N \rightarrow A\tilde{X}'_N$ induces an isomorphism on the fundamental groups.

§ 5. k -SIMPLE ACYCLIC MAPS

In this section we study acyclic maps having simplicity properties. The first proposition generalizes some results of Dror [D1, Lemma 3.4].

(5.1) **PROPOSITION.** *Let $f: X \rightarrow Y$ be a map of path connected spaces with $\pi_1(f)$ an isomorphism, and let N be a perfect normal subgroup of $\pi_1(X) = \pi$. If f induces an isomorphism $H_*(X, \mathbf{Z}[\pi/N]) \xrightarrow{\sim} H_*(Y, \mathbf{Z}[\pi/N])$ and an isomorphism $\pi_i(X) \xrightarrow{\sim} \pi_i(Y)$ for $i \leq k - 1$, then*

- (1) $\pi_k(f): \pi_k(X) \rightarrow \pi_k(Y)$ is an epimorphism when N acts trivially on $\pi_k(Y)$, and
- (2) $\pi_k(f): \pi_k(X) \rightarrow \pi_k(Y)$ is an isomorphism when N acts trivially on $\pi_k(X)$ and $\pi_k(Y)$.

Proof. Let $F \rightarrow \tilde{X}_N$ be the homotopy fibre of the covering map $\tilde{f}: \tilde{X}_N \rightarrow \tilde{Y}_N$. By hypothesis it follows easily that \tilde{f} induces an isomorphism on integral homology and on $\pi_i(X) \rightarrow \pi_i(Y)$ for $i \leq k - 1$. From the Serre spectral sequence we have $H_0(\tilde{Y}_N, H_{k-1}(F)) = H_0(N, H_{k-1}(F)) = 0$. Since $H_{k-1}(F) = \pi_{k-1}(F)$ is a quotient of $\pi_k(Y)$ on which the perfect group N acts trivially, it follows that $\pi_{k-1}(F) = 0$, which proves (1).

Under the hypothesis of (2) we have $\pi_i(F) = 0$ for $i < k$ and $H_0(\tilde{Y}_N, H_k(F)) = H_0(N, \pi_k(F)) = 0$. Since N acts trivially on $\pi_k(X)$ the induced morphism $\pi_k(F) \rightarrow \pi_k(X)$ must be trivial, which proves the proposition.

The following lemma, proved in [D2, Lemma 2.6], follows easily from the homology exact sequence.

$$H_1(G, M'') \rightarrow H_0(G, M') \rightarrow H_0(G, M) \rightarrow H_0(G, M'') \rightarrow 0$$

(5.2) LEMMA. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence of $\mathbf{Z}[G]$ -modules where G is a perfect group. Then M' and M'' are trivial G -modules if and only if M is a trivial G -module.

(5.3) DEFINITION. A space X is k -simple provided $\pi_1(X)$ acts trivially on $\pi_k(X)$. A map $f: X \rightarrow Y$ is k -simple provided $\ker \pi_1(f) \subset \pi_1(X)$ acts trivially on $\pi_k(X)$.

(5.4) PROPOSITION. Let $f: X \rightarrow Y$ be a map with homotopy fibre A where $\pi_1(A)$ is perfect. Then f is k -simple if and only if A is k -simple.

Proof. In the homotopy exact sequence of any fibration

$$\pi_{k+1}(Y) \rightarrow \pi_k(A) \rightarrow \pi_k(X) \rightarrow \pi_k(Y),$$

see the appendix, $\pi_1(A)$ acts trivially on $\text{im}(\pi_{k+1}(Y) \rightarrow \pi_k(A)) = M'$. If f is k -simple, then $\text{im}(\pi_1(A)) = \ker(\pi_1(f))$ acts trivially on $\pi_k(X)$. Hence $\pi_1(A)$ acts trivially on $M' \subset \pi_k(A)$ and on the quotient $\pi_k(A)/M'$. By (5.2), it acts trivially on $\pi_k(A)$.

Conversely, $\ker(\pi_1(f))$ acts trivially on $\ker(\pi_k(f)) \subset \pi_k(X)$ and trivially on $\pi_k(Y) \supset \text{im}(\pi_k(f))$. By (5.2), $\ker(\pi_1(f))$ acts trivially on $\pi_1(X)$. This proves the proposition.

(5.5) Notations. For a path connected space X and a perfect normal subgroup N of $\pi_1(X)$, we consider the following conditions:

(P_k). The group N acts trivially on $\pi_i(X)$ for $i \leq k$.

(H_k). The group N acts trivially on $H_i(\tilde{X})$ for $i \leq k$.

(5.6) PROPOSITION. For all natural numbers k we have that P_k implies H_k and H_k implies P_{k-1} . In particular, H_∞ and P_∞ are equivalent.

Proof. Consider the following commutative diagram where the rows and columns are fibrations.

$$\begin{array}{ccccccc}
 T & \longrightarrow & \tilde{AX}_N & \longrightarrow & A(BN) & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \tilde{X} & \longrightarrow & \tilde{X}_N & \longrightarrow & BN & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 F & \longrightarrow & \tilde{X}_N^+ & \longrightarrow & BN^+ & &
 \end{array}$$

By (5.4) condition P_k implies that $\pi_1(\tilde{AX}_N)$ acts trivially on $\pi_i(\tilde{AX}_N)$ for $i \leq k$. Since \tilde{AX}_N and $A(BN)$ are both acyclic and $\pi_1(\tilde{AX}_N) \xrightarrow{\sim} \pi_1(A(BN))$ is an isomorphism (by (4.4)), we deduce using (5.1) that $\pi_i(\tilde{AX}_N) \rightarrow \pi_i(A(BN))$ is an isomorphism for $i \leq k$. Thus $\pi_i(T) = 0$ for $i \leq k - 1$ and $\pi_k(T)$ is a trivial module $\pi_1(A(BN))$ since it is a quotient of $\pi_{k+1}(A(BN))$. On the other hand, we have $H_0(\pi_1(A(BN)), \pi_k(T)) = 0$ since $\pi_k(T) = H_k(T)$ and thus $H_*(\tilde{AX}_N) \rightarrow H_*(A(BN))$ is an isomorphism. Therefore, $\pi_k(T) = 0$ and $H_i(\tilde{X}) \rightarrow H_i(F)$ is an isomorphism for $i \leq k$. Hence P_k implies H_k since N acts trivially on $H_*(F)$.

Next, assume H_k holds. Then $H_i(\tilde{X}) \rightarrow H_i(F)$ is an isomorphism for $i \leq k$ by the comparision theorem for spectral sequences of fibrations with trivial actions. Since $\pi_1(F)$ is abelian, $\pi_1(F) = 0$ and $\pi_i(T) = 0$ for $i \leq k - 1$. Hence $\pi_1(\tilde{AX}_N)$ acts trivially on $\pi_i(\tilde{AX}_N)$ for $i \leq k - 1$. Using (5.4), we deduce P_{k-1} and the proposition.

(5.7) THEOREM. *Let $f: X \rightarrow Y$ be an acyclic map between CW-spaces which is k -simple for all $k \geq 2$ with $N = \ker \pi_1(f)$. Then the following is a fiber sequence*

$$\tilde{X} \rightarrow Y \xrightarrow{\alpha'} [B\pi_1(X)]_N^+$$

where α' is induced by $\alpha: X \rightarrow B\pi_1(X)$ as in (3.1) and $\pi_1(\alpha)$ is the identity.

Proof. As in the previous proposition, we have a diagram of fibrations

$$\begin{array}{ccccc}
 T & \longrightarrow & AX_N & \longrightarrow & A(BN) \\
 \downarrow & & \downarrow & & \downarrow \\
 \tilde{X} & \longrightarrow & \tilde{X}_N & \longrightarrow & BN \\
 \downarrow & & \downarrow & & \downarrow \\
 F & \longrightarrow & \tilde{Y} & \longrightarrow & BN^+ \\
 & \searrow & \downarrow & & \downarrow \\
 & & Y & \longrightarrow & [B\pi_1(X)]_N^+
 \end{array}$$

We prove $\tilde{X} \rightarrow F$ is a homotopy equivalence with the same argument used in (5.6) to show P_k implies H_k . Since F is also the fibre of $X_N^+ \rightarrow [B\pi_1(X)]_N^+$ we have proved the theorem.

(5.8) *Remark.* Using (5.1), we see that for an acyclic map $f: X \rightarrow Y$ which is k -simple for all $k \geq 2$, the homotopy groups $\pi_*(Y)$ can be computed in terms of $\pi_*(X)$ and $\pi_*(B\pi_1(X)_N^+) \cong \pi_*(BN)^+$ for $i \geq 2$. Some computations of $\pi_*(BN^+)$ for a certain perfect group N can be found for instance in [H, Chapter 7].

§ 6. ACYCLIC MAPS INTO A GIVEN SPACE

In this section we study acyclic maps $f: X \rightarrow Y$ into a fixed space Y . Two such map $f: X \rightarrow Y$ and $f': X' \rightarrow Y$ are called equivalent provided there is a homotopy equivalence $h: X \rightarrow X'$ with $f \simeq f'h$. Let $AC(Y)$ denote the class of equivalence classes of acyclic $f: X \rightarrow Y$ over Y where X and Y are CW -spaces.

(6.1) **DEFINITION.** *An extension data over a space Y is a triple (Φ, i, ϕ) where*

- (a) Φ is an extension $1 \rightarrow N \rightarrow G \rightarrow \pi_1(Y) \rightarrow 1$ with N perfect,
- (b) $i: BG \rightarrow BG_N^+$ is an acyclic map with $\ker(\pi_1(i)) = N$ (whose equivalence class is well defined by (3.5)), and
- (c) $\phi: Y \rightarrow BG_N^+$ is a 2-connected map.

Two triples of extension data (Φ, i, ϕ) and (Φ', i', ϕ') are called equivalent provided there exists an isomorphism $g: G \rightarrow G'$ making the following diagrams commutative (up to homotopy for the second one).

$$\begin{array}{ccccc}
 & & BG & \xrightarrow{Bg} & BG' \\
 & & i \downarrow & & \downarrow i' \\
 & & BG_N^+ & \xrightarrow{Bg^+} & B(G')_N^+ \\
 G & \xrightarrow{g} & G' & & \\
 & \searrow & \swarrow & & \\
 & & \pi_1(Y) & & \\
 & & & \nearrow \phi & \swarrow \phi' \\
 & & & Y &
 \end{array}$$