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§ 4. The homotopy fibre of the plus construction

(4.1) Theorem. *Let* $u : AX \to X$ *be the fibre of* $X \to X^+$ *for a* CW*-space* X. *Then for any map* $f : W \to X$ *from an acyclic* CW*-space* W *into* X, *there is a map* $f' : W \to AX$ *with* $uf' \simeq f$ *and* f *is unique up to homotopy.*

Proof. We have the following diagram where the lower row is a fibre sequence.

$$\begin{array}{ccccccc} & & & & W & & \\ & & & \swarrow^{f'} & \downarrow f & & \\ \Omega(X^+) & \longrightarrow & AX & \xrightarrow{u} & X & \xrightarrow{\theta} & X^+ \end{array}$$

Since $\pi_1(W)$ is perfect and $\pi_1(X^+)$ contains no nonzero perfect subgroups, $\pi_1(\theta f)$ is zero and by (3.2) the map θf is null homotopic. Then there is a map $f' : W \to AX$ with $uf' \simeq f$. Two factorizations f' of f differ by the action of a map $W \to \Omega(X^+)$. Since again $\pi_1(W)$ is perfect and $\pi_1(\Omega(X^+))$ abelian, π_1 of this map is zero so by (3.2) the map is null homotopic. Hence f' is unique, and this proves the theorem.

(4.2) *Remark.* Dror introduced the map $AX \to X$ having the universal property given in the previous theorem and proved for each CW-space X the map $AX \to X$ existed. He used a Posnikov tower construction starting with the covering of X corresponding to the maximal perfect normal subgroup of $\pi_1(X)$. By (2.5) we see that we can recover $X \to X^+$ as the cofibre of $AX \to X$.

All the properties of AX listed in [D1, Theorem 2.1] can be shown using the fact that AX is the fibre of $X \to X^+$. For instance we will in (5.4) give a sharper version of [D1, Theorem 2.1 (iv)].

(4.3) *Remark.* The Posnikov tower construction for $AX \to X$, when done in the category of simplicial sets, is functorial for maps of simplicial sets. For CW-spaces we obtain a functorial $AX \to X$ for maps using the geometric realization of simplicial sets. Since we can choose $X \to X_N^+$ to be the cofibre of $A(\tilde{X}_N) \to X$, we obtain a sharper version of the functoriality in (3.7) and (3.8), namely on the level of spaces and maps.

(4.4) *Remark.* The group $\tilde{N} = \pi_1 (A\tilde{X}_N)$ is a central extension of N (see the appendix) and, as $A\tilde{X}_N$ is acyclic, satisfies $H_1 (\tilde{N}) = H_2 (\tilde{N}) = 0$. Therefore $\tilde{N}$ is the universal central extension of N (see [K2]), namely one has the exact sequence $0 \to H_2 (N) \to \tilde{N} \to N \to 1$. Therefore, if $f: X \to X'$ is a map such that $\pi_1 (f)$ sends the perfect normal subgroup N of $\pi_1 (X)$ isomorphically onto a normal subgroup N' of $\pi_1 (X')$, then the induced map $Af: A\tilde{X}_N \to A\tilde{X}'_N$ induces an isomorphism on the fundamental groups.

§ 5. k-SIMPLE ACYCLIC MAPS

In this section we study acyclic maps having simplicity properties. The first proposition generalizes some results of Dror [D1, Lemma 3.4].

(5.1) PROPOSITION. *Let $f: X \to Y$ be a map of path connected spaces with $\pi_1 (f)$ an isomorphism, and let N be a perfect normal subgroup of $\pi_1 (X) = \pi$. If f induces an isomorphism $H_* (X, \mathbf{Z} [\pi/N]) \xrightarrow{\sim} H_* (Y, \mathbf{Z} [\pi/N])$ and an isomorphism $\pi_i (X) \xrightarrow{\sim} \pi_i (Y)$ for $i \leqq k - 1$, then*

(1) *$\pi_k (f) : \pi_k (X) \to \pi_k (Y)$ is an epimorphism when N acts trivially on $\pi_k (Y)$, and*

(2) *$\pi_k (f) : \pi_k (X) \to \pi_k (Y)$ is an isomorphism when N acts trivially on $\pi_k (X)$ and $\pi_k (Y)$.*

Proof. Let $F \to \tilde{X}_N$ be the homotopy fibre of the covering map $\tilde{f}: \tilde{X}_N \to \tilde{Y}_N$. By hypothesis it follows easily that $\tilde{f}$ induces an isomorphism on integral homology and on $\pi_i (X) \to \pi_i (Y)$ for $i \leqq k - 1$. From the Serre spectral sequence we have $H_0 (\tilde{Y}_N, H_{k-1} (F)) = H_0 (N, H_{k-1} (F)) = 0$. Since $H_{k-1} (F) = \pi_{k-1} (F)$ is a quotient of $\pi_k (Y)$ on which the perfect group N acts trivially, it follows that $\pi_{k-1} (F) = 0$, which proves (1).

Under the hypothesis of (2) we have $\pi_i (F) = 0$ for $i < k$ and $H_0 (\tilde{Y}_N, H_k (F)) = H_0 (N, \pi_k (F)) = 0$. Since N acts trivially on $\pi_k (X)$ the induced morphism $\pi_k (F) \to \pi_k (X)$ must be trivial, which proves the proposition.

The following lemma, proved in [D2, Lemma 2.6], follows easily from the homology exact sequence.