

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 25 (1979)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ACYCLIC MAPS
Autor: Hausmann, Jean-Claude / Husemoller, Dale
Kapitel: §4. The homotopy fibre of the plus construction
DOI: <https://doi.org/10.5169/seals-50372>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 07.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

§ 4. THE HOMOTOPY FIBRE OF THE PLUS CONSTRUCTION

(4.1) THEOREM. *Let $u : AX \rightarrow X$ be the fibre of $X \rightarrow X^+$ for a CW-space X . Then for any map $f : W \rightarrow X$ from an acyclic CW-space W into X , there is a map $f' : W \rightarrow AX$ with $uf' \simeq f$ and f' is unique up to homotopy.*

Proof. We have the following diagram where the lower row is a fibre sequence.

$$\begin{array}{ccccc} & & W & & \\ & \swarrow f' & \downarrow f & & \\ \Omega(X^+) & \longrightarrow & AX & \xrightarrow{u} & X \xrightarrow{\theta} X^+ \end{array}$$

Since $\pi_1(W)$ is perfect and $\pi_1(X^+)$ contains no nonzero perfect subgroups, $\pi_1(\theta f)$ is zero and by (3.2) the map θf is null homotopic. Then there is a map $f' : W \rightarrow AX$ with $uf' \simeq f$. Two factorizations f' of f differ by the action of a map $W \rightarrow \Omega(X^+)$. Since again $\pi_1(W)$ is perfect and $\pi_1(\Omega(X^+))$ abelian, π_1 of this map is zero so by (3.2) the map is null homotopic. Hence f' is unique, and this proves the theorem.

(4.2) *Remark.* Dror introduced the map $AX \rightarrow X$ having the universal property given in the previous theorem and proved for each CW-space X the map $AX \rightarrow X$ existed. He used a Postnikov tower construction starting with the covering of X corresponding to the maximal perfect normal subgroup of $\pi_1(X)$. By (2.5) we see that we can recover $X \rightarrow X^+$ as the cofibre of $AX \rightarrow X$.

All the properties of AX listed in [D1, Theorem 2.1] can be shown using the fact that AX is the fibre of $X \rightarrow X^+$. For instance we will in (5.4) give a sharper version of [D1, Theorem 2.1 (iv)].

(4.3) *Remark.* The Postnikov tower construction for $AX \rightarrow X$, when done in the category of simplicial sets, is functorial for maps of simplicial sets. For CW-spaces we obtain a functorial $AX \rightarrow X$ for maps using the geometric realization of simplicial sets. Since we can choose $X \rightarrow X_N^+$ to be the cofibre of $A(\tilde{X}_N) \rightarrow X$, we obtain a sharper version of the functoriality in (3.7) and (3.8), namely on the level of spaces and maps.

(4.4) *Remark.* The group $\tilde{N} = \pi_1(A\tilde{X}_N)$ is a central extension of N (see the appendix) and, as $A\tilde{X}_N$ is acyclic, satisfies $H_1(\tilde{N}) = H_2(\tilde{N}) = 0$. Therefore \tilde{N} is the universal central extension of N (see [K2]), namely one has the exact sequence $0 \rightarrow H_2(N) \rightarrow \tilde{N} \rightarrow N \rightarrow 1$. Therefore, if $f: X \rightarrow X'$ is a map such that $\pi_1(f)$ sends the perfect normal subgroup N of $\pi_1(X)$ isomorphically onto a normal subgroup N' of $\pi_1(X')$, then the induced map $Af: A\tilde{X}_N \rightarrow A\tilde{X}'_N$ induces an isomorphism on the fundamental groups.

§ 5. k -SIMPLE ACYCLIC MAPS

In this section we study acyclic maps having simplicity properties. The first proposition generalizes some results of Dror [D1, Lemma 3.4].

(5.1) **PROPOSITION.** *Let $f: X \rightarrow Y$ be a map of path connected spaces with $\pi_1(f)$ an isomorphism, and let N be a perfect normal subgroup of $\pi_1(X) = \pi$. If f induces an isomorphism $H_*(X, \mathbf{Z}[\pi/N]) \xrightarrow{\sim} H_*(Y, \mathbf{Z}[\pi/N])$ and an isomorphism $\pi_i(X) \xrightarrow{\sim} \pi_i(Y)$ for $i \leq k - 1$, then*

- (1) $\pi_k(f): \pi_k(X) \rightarrow \pi_k(Y)$ is an epimorphism when N acts trivially on $\pi_k(Y)$, and
- (2) $\pi_k(f): \pi_k(X) \rightarrow \pi_k(Y)$ is an isomorphism when N acts trivially on $\pi_k(X)$ and $\pi_k(Y)$.

Proof. Let $F \rightarrow \tilde{X}_N$ be the homotopy fibre of the covering map $\tilde{f}: \tilde{X}_N \rightarrow \tilde{Y}_N$. By hypothesis it follows easily that \tilde{f} induces an isomorphism on integral homology and on $\pi_i(X) \rightarrow \pi_i(Y)$ for $i \leq k - 1$. From the Serre spectral sequence we have $H_0(\tilde{Y}_N, H_{k-1}(F)) = H_0(N, H_{k-1}(F)) = 0$. Since $H_{k-1}(F) = \pi_{k-1}(F)$ is a quotient of $\pi_k(Y)$ on which the perfect group N acts trivially, it follows that $\pi_{k-1}(F) = 0$, which proves (1).

Under the hypothesis of (2) we have $\pi_i(F) = 0$ for $i < k$ and $H_0(\tilde{Y}_N, H_k(F)) = H_0(N, \pi_k(F)) = 0$. Since N acts trivially on $\pi_k(X)$ the induced morphism $\pi_k(F) \rightarrow \pi_k(X)$ must be trivial, which proves the proposition.

The following lemma, proved in [D2, Lemma 2.6], follows easily from the homology exact sequence.