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§ 4. THE HOMOTOPY FIBRE OF THE PLUS CONSTRUCTION

(4.1) THEOREM. *Let  $u : AX \rightarrow X$  be the fibre of  $X \rightarrow X^+$  for a CW-space  $X$ . Then for any map  $f : W \rightarrow X$  from an acyclic CW-space  $W$  into  $X$ , there is a map  $f' : W \rightarrow AX$  with  $uf' \simeq f$  and  $f'$  is unique up to homotopy.*

*Proof.* We have the following diagram where the lower row is a fibre sequence.

$$\begin{array}{ccccccc}
 & & & W & & & \\
 & & & \downarrow f & & & \\
 & & s' & & & & \\
 & & \swarrow & & & & \\
 \Omega(X^+) & \longrightarrow & AX & \xrightarrow{u} & X & \xrightarrow{\theta} & X^+
 \end{array}$$

Since  $\pi_1(W)$  is perfect and  $\pi_1(X^+)$  contains no nonzero perfect subgroups,  $\pi_1(\theta f)$  is zero and by (3.2) the map  $\theta f$  is null homotopic. Then there is a map  $f' : W \rightarrow AX$  with  $uf' \simeq f$ . Two factorizations  $f'$  of  $f$  differ by the action of a map  $W \rightarrow \Omega(X^+)$ . Since again  $\pi_1(W)$  is perfect and  $\pi_1(\Omega(X^+))$  abelian,  $\pi_1$  of this map is zero so by (3.2) the map is null homotopic. Hence  $f'$  is unique, and this proves the theorem.

(4.2) Remark. Dror introduced the map  $AX \rightarrow X$  having the universal property given in the previous theorem and proved for each CW-space  $X$  the map  $AX \rightarrow X$  existed. He used a Posnikov tower construction starting with the covering of  $X$  corresponding to the maximal perfect normal subgroup of  $\pi_1(X)$ . By (2.5) we see that we can recover  $X \rightarrow X^+$  as the cofibre of  $AX \rightarrow X$ .

All the properties of  $AX$  listed in [D1, Theorem 2.1] can be shown using the fact that  $AX$  is the fibre of  $X \rightarrow X^+$ . For instance we will in (5.4) give a sharper version of [D1, Theorem 2.1 (iv)].

(4.3) Remark. The Posnikov tower construction for  $AX \rightarrow X$ , when done in the category of simplicial sets, is functorial for maps of simplicial sets. For CW-spaces we obtain a functorial  $AX \rightarrow X$  for maps using the geometric realization of simplicial sets. Since we can choose  $X \rightarrow X_N^+$  to be the cofibre of  $A(\tilde{X}_N) \rightarrow X$ , we obtain a sharper version of the functoriality in (3.7) and (3.8), namely on the level of spaces and maps.

(4.4) *Remark.* The group  $\tilde{N} = \pi_1(A\tilde{X}_N)$  is a central extension of  $N$  (see the appendix) and, as  $A\tilde{X}_N$  is acyclic, satisfies  $H_1(\tilde{N}) = H_2(\tilde{N}) = 0$ . Therefore  $\tilde{N}$  is the universal central extension of  $N$  (see [K2]), namely one has the exact sequence  $0 \rightarrow H_2(N) \rightarrow \tilde{N} \rightarrow N \rightarrow 1$ . Therefore, if  $f: X \rightarrow X'$  is a map such that  $\pi_1(f)$  sends the perfect normal subgroup  $N$  of  $\pi_1(X)$  isomorphically onto a normal subgroup  $N'$  of  $\pi_1(X')$ , then the induced map  $Af: A\tilde{X}_N \rightarrow A\tilde{X}'_N$  induces an isomorphism on the fundamental groups.

### § 5. $k$ -SIMPLE ACYCLIC MAPS

In this section we study acyclic maps having simplicity properties. The first proposition generalizes some results of Dror [D1, Lemma 3.4].

(5.1) PROPOSITION. *Let  $f: X \rightarrow Y$  be a map of path connected spaces with  $\pi_1(f)$  an isomorphism, and let  $N$  be a perfect normal subgroup of  $\pi_1(X) = \pi$ . If  $f$  induces an isomorphism  $H_*(X, \mathbf{Z}[\pi/N]) \xrightarrow{\sim} H_*(Y, \mathbf{Z}[\pi/N])$  and an isomorphism  $\pi_i(X) \xrightarrow{\sim} \pi_i(Y)$  for  $i \leq k - 1$ , then*

- (1)  $\pi_k(f): \pi_k(X) \rightarrow \pi_k(Y)$  is an epimorphism when  $N$  acts trivially on  $\pi_k(Y)$ , and
- (2)  $\pi_k(f): \pi_k(X) \rightarrow \pi_k(Y)$  is an isomorphism when  $N$  acts trivially on  $\pi_k(X)$  and  $\pi_k(Y)$ .

*Proof.* Let  $F \rightarrow \tilde{X}_N$  be the homotopy fibre of the covering map  $\tilde{f}: \tilde{X}_N \rightarrow \tilde{Y}_N$ . By hypothesis it follows easily that  $\tilde{f}$  induces an isomorphism on integral homology and on  $\pi_i(X) \rightarrow \pi_i(Y)$  for  $i \leq k - 1$ . From the Serre spectral sequence we have  $H_0(\tilde{Y}_N, H_{k-1}(F)) = H_0(N, H_{k-1}(F)) = 0$ . Since  $H_{k-1}(F) = \pi_{k-1}(F)$  is a quotient of  $\pi_k(Y)$  on which the perfect group  $N$  acts trivially, it follows that  $\pi_{k-1}(F) = 0$ , which proves (1).

Under the hypothesis of (2) we have  $\pi_i(F) = 0$  for  $i < k$  and  $H_0(\tilde{Y}_N, H_k(F)) = H_0(N, \pi_k(F)) = 0$ . Since  $N$  acts trivially on  $\pi_k(X)$  the induced morphism  $\pi_k(F) \rightarrow \pi_k(X)$  must be trivial, which proves the proposition.

The following lemma, proved in [D2, Lemma 2.6], follows easily from the homology exact sequence.