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§ 2. INDUCED AND COINDUCED ACYCLIC MAPS

(2.1) PROPOSITION. Let $f: X \to Y$ and $g: Y \to Z$ be two maps. If f and g are acyclic, then gf is acyclic. If f and gf are acyclic, then g is acyclic.

Proof. Consider a local system L on Z, and using g^*L on $Y f^*g^*L = (gf)^*L$ on X, we apply (1.2) (b) to obtain the proposition.

(2.2) **PROPOSITION.** Consider the following cartesian square where either f or g is a fibration.

If f is acyclic, then f' is acyclic.

Proof. Since either f or g is a fibration, we can change the other to be a fibration, if necessary, without changing the homotopy type of any of the four spaces. Now the homotopy fibre F of f is the actual fiber and F is also the homotopy fibre of f'. Now apply (1.2) (a).

(2.3) PROPOSITION. Consider the following cocartesian square where either f or g is a cofibration.

If f is acyclic, then f' is acyclic.

Proof. Since either f or g is a cofibration, we can change the other to be a cofibration, if necessary, without changing the homotopy type of any of the four spaces. Hence each map is an injection, and for a local coefficient system L on Y', we have two long exact sequences in homology

$$\longrightarrow H_q(X, f^*g'^*L) \xrightarrow{f_*} H_q(Y, f'^*L) \longrightarrow H_q(Y, X; f'^*L) \longrightarrow \dots$$

$$\downarrow g_* \qquad \qquad \downarrow g'_* \qquad \qquad \downarrow (g, g')_*$$

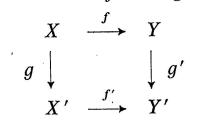
$$\longrightarrow H_q(X', g'^*L) \xrightarrow{f'_*} H_q(Y', L) \longrightarrow H_q(Y', X'; L) \longrightarrow \dots$$

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By hypothesis (1.2) (b) the morphism f_* is an isomorphism and thus $H_*(Y, X; f'^*L) = 0$. By excision $(g, g')_*$ is an isomorphism and thus $H_*(Y', X'; L) = 0$. Hence f'_* is an isomorphism and criterion (1.2) (b) is satisfied for f' to be an acyclic map which proves the proposition.

The previous proposition concerning acyclic maps in a cofibration will be the basic tool for most of the results which follow in sections 2 and 3. It was pointed out to us by Quillen.

(2.4) PROPOSITION. Consider the following diagram of CW-spaces.



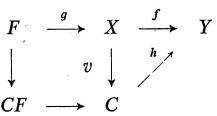
If g and g' are acyclic, and if $\pi_1(f)$ and $\pi_1(f')$ are isomorphisms then the diagram is cocartesian up to homotopy equivalence.

Proof. First replace f and g by equivalent cofibrations and form $h: X' \cup_X Y \to Y'$. The map $g'': Y \to X' \cup_X Y$ is an acyclic map by (2.3) and g' = hg''. Thus h is acyclic by (2.1).

Since $\pi_1(f)$ is an isomorphism, it follows that $f'': X' \to X' \cup {}_X Y$ has the property that $\pi_1(f'')$ is an isomorphism by the van Kampen theorem and f' = hf''. Thus $\pi_1(h)$ is an isomorphism. Now apply (1.5) to see that his a homotopy equivalence. This proves the proposition.

(2.5) THEOREM. Let $f: X \to Y$ be an acyclic map between CW-spaces with homotopy fibre $g: F \to X$. Then f is the homotopy cofibre of g.

Proof. Let CF be the cone over F. The homotopy cofibre C of $g: F \to X$ is homotopy equivalent to $CF \cup {}_{F}X$ and we have the cocartesian square



Since $fg \simeq *$, it follows that we have a map $h: C \to Y$ such that $f \simeq hv$. Since f is acyclic, the map $F \to CF$ is acyclic and, by (2.3) v is acyclic. One deduces then, by (2.1) that h is acyclic. As π_1 (h) is onto (1.3), one has:

 $\ker \left(\pi_1(h)\right) = v\left(\ker \pi_1(f)\right) = v\left(\operatorname{Im} \pi_1(g)\right) = 1$

So $\pi_1(h)$ is injective and, by (1.3) and (1.5), h is a homotopy equivalence.

(2.6) THEOREM. Let $f: X \to Y$ be an acyclic map between CW-spaces and let $h_1, h_2: Y \to Z$ be two maps. If $h_1 f \simeq h_2 f$, then it follows that $h_1 \simeq h_2$.

Proof. By (2.5) we have cofibre sequence

 $F \xrightarrow{g} X \xrightarrow{f} Y \longrightarrow \Delta F$

where ΔF is the reduced suspension of the acyclic space F. Since ΔF is simply connected and $H_*(\Delta F) = 0$, it is contractible, and the group $[\Delta F, Z]$ in the Puppe sequence is zero.

In general, the group $[\Delta F, Z]$ acts transitively on the fibres of the function $[Y, Z] \rightarrow [X, Z]$, so that in this case, $[Y, Z] \rightarrow [X, Z]$ is injective. This proves the theorem.

§ 3. CLASSIFICATION OF ACYCLIC MAP FROM A GIVEN SPACE

Let X be a path connected space. To each acyclic map $f: X \to Y$, we assign the kernel of $\pi_1(f): \pi_1(X) \to \pi_1(Y)$ which is a perfect normal subgroup of $\pi_1(X)$ by (1.3). The object of this section is to show that this map from isomorphism classes of acyclic maps defined on X to perfect normal subgroups of $\pi_1(X)$ is a bijection.

(3.1) PROPOSITION. Let $f: X \to Y$ and $f': X \to Y'$ be two maps between CW-spaces such that f is acyclic. There exists a map $h: Y \to Y'$ with $hf \simeq f'$ if and only if ker $\pi_1(f) \subset \ker \pi_1(f')$, and such an h is unique up to homotopy. In addition, if f' is acyclic, then h is acyclic, and his a homotopy equivalence if and only if ker $\pi_1(f) = \ker \pi_1(f')$.

Proof. If h exists, then $\pi_1(f') = \pi_1(h) \circ \pi_1(f)$ and we have ker $\pi_1(f) \subset \ker \pi_1(f')$. Conversely, we can suppose f is a cofibration and form the cocartesian diagram