Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 25 (1979)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: RECENT PROGRESS IN THE THEORY OF MINIMAL SURFACES

Autor: Bombieri, E.

Kapitel: V. Recent progress on regularity problems

DOI: https://doi.org/10.5169/seals-50368

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 03.12.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

u = v. This corresponds to the well-known cone $x_1^2 + x_2^2 = x_3^2 + x_4^2$, which has indeed mean curvature 0. The other possibility for a geodesic is to end on the u-axis, those ending on the v-axis being obtained by a symmetrical reflection. Up to a homothetic transformation there is only one such geodesic. We introduce the new homothetically invariant parameters

$$\varphi = \operatorname{artg} \frac{v}{u}, \quad \theta = \operatorname{artg} \frac{v'}{u'},$$

$$\sigma = \theta - 3\varphi + \frac{\pi}{2}, \quad \psi = \theta + \varphi - \frac{\pi}{2}$$

and rewrite the equation for geodesics as

$$\begin{cases} \dot{\sigma} = -\frac{3}{2} \sin \sigma - \frac{7}{2} \sin \psi \\ \dot{\psi} = \frac{1}{2} \sin \sigma - \frac{3}{2} \sin \psi \end{cases}$$

We are interested in the unique characteristic C which at time $t=-\infty$ starts at the saddle point $(\pi,0)$ and at time $t=\infty$ ends at the origin (0,0). Since the diagonal u=v goes in the line $\sigma=\psi$ in the (σ,ψ) -plane, if we follow C from $t=-\infty$ to a time t_0 for which $\sigma=\psi$, going back to the (u,v) plane we get a geodesic starting on the axis v=0 and ending on u=v; clearly by applying a suitable homothety we may get a geodesic ending at $u=v=\frac{1}{\sqrt{2}}$ and a solution to our problem. It follows that our result will be proved if we show that the characteristic C crosses the line $\sigma=\psi$ infinitely many times. This in fact is obvious, because C ends at (0,0) and it is easily checked that (0,0) is a focal singular point, or vortex, of the differential system for σ,ψ .

It may be noted that the same construction gives other examples, like for the boundary $S^2\left(\frac{1}{\sqrt{2}}\right)\times S^2\left(\frac{1}{\sqrt{2}}\right)$, with almost exactly the same result.

V. RECENT PROGRESS ON REGULARITY PROBLEMS

The regularity theory of minimal currents and varifolds is fundamental if we want to obtain classical solutions to variational problems. Here the theory proceeds in two main directions: one is to prove stronger and better

regularity theorems, the other is to produce more examples of singular minimal varieties to narrow the gap.

It is a classical result that a minimal surface is real analytic at every regular point. Let $V \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ be a complex analytic subvariety of \mathbb{C}^n ; by Wirtinger's inequality, V is also an absolutely minimizing surface in \mathbb{R}^{2n} , hence V may carry singularities and the singular set can have codimension 2. If T is a minimal hypersurface in \mathbb{R}^n , singularities are harder to find: Simons' cone

$$x_1^2 + \dots + x_4^2 = x_5^2 + \dots + x_8^2$$

is the first and simplest example of a singular absolutely minimal hypersurface in \mathbb{R}^8 . All these examples are real analytic sets and one could ask whether this is always the case. However there are topological obstructions for a singularity to be real analytic, as the following construction by Milani shows.

We can find an embedding of $\mathbf{P}^2(\mathbf{C})$, the complex projective plane, in \mathbf{R}^9 so that $\mathbf{P}^2(\mathbf{C})$ will be on the sphere S^8 given by $x_1^2 + ... + x_9^2 = 1$. By the general theory, there is a 5-dimensional current T with boundary $\mathbf{P}^2(\mathbf{C})$ which is absolutely minimizing and, by results of Allard on boundary regularity, one can show that spt (T) is a manifold in a neighborhood of its boundary. We conclude that the singular set of T is a compact subset of spt $(T) \setminus \text{spt } (\partial T)$. Now assume that spt (T) is a real analytic set Σ . By Hironaka's theorem on resolution of singularities, together with a very important refinement obtained by Tognoli, there is a real analytic manifold Σ' and a proper $f: \Sigma' \to \Sigma$ which is an isomorphism outside $f^{-1}(\sin \Sigma)$. Thus Σ' is a real manifold with boundary $\mathbf{P}^2(\mathbf{C})$. This contradicts Thom's theorem that $\mathbf{P}^2(\mathbf{C})$ is a generator of infinite order of the cobordism ring, and the conclusion is that spt (T) is not a real analytic set.

Another beautiful example has been obtained by Lawson and Osserman [L-O] in their work on the Dirichlet problem on the minimal surface system in non-parametric form, in higher codimension. If $\eta: S^3 \to S^2$ is the Hopf map

$$\eta(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1z_2)$$

where $(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \cong \mathbb{R}^4$ and η is considered as $\eta(z_1, z_2) \in \mathbb{R} \times \mathbb{C}$ $\cong \mathbb{R}^3$, they found that the Lipschitz function $f: \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$f(x) = \frac{\sqrt{5}}{2} |x| \eta\left(\frac{x}{|x|}\right) \quad \text{for } x \neq 0,$$

is a solution of the minimal surface system. This gives the first example of a non-parametric minimal Lipschitz cone, of dimension 4 in \mathbb{R}^7 .

General regularity theorems for absolutely minimal currents have proved to be very difficult to obtain. The codimension 1 case has been treated with success; after previous work by Reifenberg, De Giorgi, Almgren, Miranda, Simons, finally Federer [FH 2] proved the sharp result that absolutely minimal hypersurfaces are non-singular in codimension less than 7. In particular, minimal hypersurfaces of dimension ≤ 6 are analytic manifolds. Also, in the codimension one non-parametric case Bombieri, De Giorgi and Miranda proved regularity in any dimension, a result to be contrasted with the Lipschitz singular cone of Lawson and Osserman.

In general codimensions, the only result was that the set of regular points is dense (Reifenberg, Morrey, Almgren) and only recently Almgren announced [AF3] that minimal surfaces are regular almost everywhere. It seems likely that Almgren's new methods will in fact show that minimal surfaces are regular in codimension 2; in view of the examples provided by complex analytic varieties, this result would be sharp.

REFERENCES

- [AF 1] Almgren, F. J., Jr. The theory of varifolds. Mimeographed notes, Princeton (1965).
- [AF 2] The homotopy groups of the integral cycle groups. *Topology 1* (1962), pp. 257-299.
- [AF 3] Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents. U.S.-Japan Seminar on Minimal Submanifolds, Including Geodesics. Tokyo 1977.
- [B-DG-G] Bombieri, E., E. De Giorgi and E. Giusti. Minimal Cones and the Bernstein problem. *Inventiones Math.* 7 (1969), pp. 243-269.
- [F-F] FEDERER, H. and W. FLEMING. Normal and integral currents. *Annals of Math.* 72 (1960), pp. 458-520.
- [FH 1] FEDERER, H. Geometric Measure Theory. Springer Verlag, New York 1969.
- [FH 2] The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension. *Bull. Am. Math. Soc.* 76 (1970), pp. 767-771.
- [L] LAWSON, H. B., Jr. The global behavior of minimal surfaces in S^n . Ann. of Math. (2) 92 (1970), pp. 224-237.
- [L-O] Lawson, H. B. Jr. and R. Osserman. Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system. *Acta Math.* 139 (1977), pp. 1-17.
- [M] Morgan, F. Almost every curve in R³ bounds a unique area minimizing surface. To appear in *Inventiones Math.* (1978).
- [P 1] PITTS, J. Existence and regularity of minimal surfaces on riemannian manifolds. *Bull. Am. Math. Soc.* 82 (1976), pp. 503-504.
- [P 2] Existence of minimal surfaces on riemannian manifolds, parts I and II, preprint.

Enrico Bombieri

(Reçu le 15 septembre 1978)

The Institute for advanced Study Princeton, N.J. (USA)