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2. INTEGRAL REPRESENTATION THEOREMS FOR LINEAR FUNCTIONALS

Let A be a commutative Banach algebra over \mathbb{C} and let Δ denote the locally compact space of regular maximal ideals of A . For each $x \in A$ we use \hat{x} to denote the Gelfand-transform; i.e., \hat{x} is the continuous mapping from Δ to \mathbb{C} defined by the relations:

$$\hat{x}(m) = m(x) \quad \text{for } m \in \Delta.$$

By $C_0(\Delta)$ we shall denote the algebra of all complex-valued continuous functions on Δ which vanish at infinity. For any subset $\mathcal{A} \subset A$ we shall use the notation $\hat{\mathcal{A}}$ to denote the set $\{\hat{x} : x \in \mathcal{A}\}$. As usual $\|\hat{x}\|_\infty$ denotes the supremum norm.

THEOREM 1. Let f be a linear form on the complex commutative Banach algebra A and let \mathcal{A} be a linear subspace of A . The following two statements are equivalent:

(1) There exists a constant M such that

$$|f(x)| \leq M \|\hat{x}\|_\infty \quad \text{for every } x \in \mathcal{A}.$$

(2) There exists a bounded complex Radon measure μ on Δ such that

$$f(x) = \int_{\Delta} \hat{x}(m) d\mu(m) \quad \text{for every } x \in \mathcal{A}.$$

Proof. The implication (2) \Rightarrow (1) is clear with $M = \|\mu\|$. We shall prove (1) \Rightarrow (2). Define a mapping $L : \hat{\mathcal{A}} \rightarrow \mathbb{C}$ by

$$L(\hat{x}) = f(x).$$

It follows from (1) that L is well-defined, and that

$$|L(\hat{x})| \leq M \|\hat{x}\|_\infty \quad \text{for every } \hat{x} \in \hat{\mathcal{A}}$$

and so L is continuous with $\|L\| \leq M$. Using the Hahn-Banach Theorem we can extend L to a bounded linear form L_0 on $C_0(\Delta)$ and by the Riesz Representation Theorem we obtain the existence of a bounded complex Radon measure μ on Δ such that

$$\|\mu\| = \|L\| = \|L_0\| \quad \text{and}$$

$$L_0(\varphi) = \int_{\Delta} \varphi(m) d\mu(m) \quad \text{for every } \varphi \in C_0(\Delta).$$

In particular

$$f(x) = L(\hat{x}) = L_0(\hat{x}) = \int_{\Delta} \hat{x}(m) d\mu(m) \quad \text{for every } x \in \mathcal{A}.$$

Remark: Suppose that A has an identity and that \hat{A} is closed under complex conjugation, then since \hat{A} contains constants and separates the points of Δ , the Stone-Weierstraß Theorem implies that \hat{A} is dense in $C(\Delta)$, the algebra of all complex-valued continuous functions on the compact Hausdorff space Δ . If we impose these additional conditions on A and if we take $\mathcal{A} = A$ in Theorem 1, we can conclude that in this case the representing measure μ is uniquely determined.

If the algebra A has a continuous involution, one can use Theorem 1 to derive an extended version of a theorem due to Raikov [10]. We proceed to describe the situation.

Let A be a complex commutative Banach algebra with an isometric involution $*$ and a bounded approximate identity $\{u_\lambda\}_{\lambda \in \Lambda}$ i.e., a net satisfying the following conditions:

$$\begin{aligned} \|u_\lambda\| &\leq 1 \quad \text{for each } \lambda \in \Lambda, \\ \|u_\lambda x - x\| &\rightarrow 0 \quad \text{for each } x \in A. \end{aligned}$$

A continuous *positive* functional on A is an element $f \in A'$ such that $f(x^*x) \geq 0$ for every $x \in A$. If f is a continuous positive functional on A then the Cauchy-Schwarz inequality is valid (Dixmier [8, p. 23]) and this implies the following facts:

$$\begin{aligned} f(u_\lambda) &\rightarrow \|f\| \\ |f(x)|^2 &\leq \|f\| f(x^*x) \quad \text{for every } x \in A. \end{aligned}$$

If the involution is *symmetric*, which means $(x^*)^\wedge = \overline{\hat{x}}$ for every $x \in A$ or, equivalently, that every $m \in \Delta$ is a *positive* linear functional, then by modifying a classical method of Gelfand-Raikov-Silov [10; p. 62] one can prove that

$$|f(x)| \leq \|f\| \|\hat{x}\|_\infty \quad \text{for every } x \in A.$$

As a corollary to Theorem 1 and the above discussion we obtain the following extended theorem of Raikov [10; p. 64], see also Bucy-Maltese [4]):

THEOREM 2. Let A be a commutative Banach algebra with an isometric involution which is symmetric. Suppose that A has a bounded approximate identity and let $f \in A'$ be a continuous positive functional. Then there exists a unique positive Radon measure μ on Δ such that $\|\mu\| = \|f\|$ and

$$f(x) = \int_A \hat{x}(m) d\mu(m) \quad \text{for every } x \in A.$$

Proof. From the above remarks we know that

$$|f(x)| \leq \|f\| \|\hat{x}\|_{\infty} \quad \text{for every } x \in A.$$

By Theorem 1 there exists a complex Radon measure μ on Δ such that $\|\mu\| \leq \|f\|$ and

$$f(x) = \int_A \hat{x}(m) d\mu(m) \quad \text{for every } x \in A.$$

This formula implies

$$|f(x)| \leq \|\hat{x}\|_{\infty} \|\mu\| \leq \|x\| \|\mu\| \quad \text{for every } x \in A,$$

so that $\|f\| \leq \|\mu\|$ and hence $\|f\| = \|\mu\|$.

Since \hat{A} is a self-adjoint subalgebra of $C_0(\Delta)$ which separates points and for each $m \in \Delta$ contains a function \hat{x} such that $\hat{x}(m) \neq 0$ (in fact ¹⁾ there exists an element u_{β} of the approximate identity such that $\hat{u}_{\beta}(m) \neq 0$), the Stone-Weierstraß Theorem implies the uniqueness of the measure μ . The positivity of μ also follows from the fact that \hat{A} is dense in $C_0(\Delta)$. In fact if p is a non-negative function in $C_0(\Delta)$, then $p = |q|^2$ for some $q \in C_0(\Delta)$. Choose a sequence $\{x_n\}$ in A such that

$$\hat{x}_n \rightarrow q.$$

¹⁾ If $m \in \Delta$, then $\|m\| \neq 0$ and by the assumption of symmetry m is a positive functional. Therefore, as mentioned above, $\|m\| = \lim_{\alpha} m(u_{\alpha})$ so that there must exist some u_{β} of the approximate identity such that $m(u_{\beta}) \neq 0$.

Since $(x_n^*)^\wedge = \overline{x_n}$ it follows that $(x_n^*)^\wedge \rightarrow \bar{q}$ and hence

$$(x_n x_n^*)^\wedge \rightarrow |q|^2 = p.$$

This implies

$$\begin{aligned} \int_A p(m) d\mu(m) &= \lim_n \int_A (x_n^* x_n)^\wedge(m) d\mu(m) \\ &= \lim_n f(x_n^* x_n) \geq 0, \end{aligned}$$

so that μ is a positive measure and this completes the proof.

If A has an identity, as is the case in Raikov's original version, the above proof can be somewhat simplified.

THEOREM 3 (Raikov). Let A be a complex commutative Banach algebra with an identity e and with an isometric involution which is symmetric. If f is a continuous positive functional on A , then there exists a unique positive Radon measure μ on Δ such that $\|\mu\| = \|f\|$ and

$$f(x) = \int_A \hat{x}(m) d\mu(m) \quad \text{for every } x \in A.$$

Proof. As above we know that

$$|f(x)| \leq \|f\| \|\hat{x}\|_\infty \quad \text{for every } x \in A.$$

From Theorem 1 there exists a complex Radon measure μ on Δ such that $\|\mu\| \leq \|f\|$ and

$$f(x) = \int_A \hat{x}(m) d\mu(m) \quad \text{for every } x \in A.$$

Hence $\|\mu\| \leq \|f\| = f(e) = \mu(1) \leq \|\mu\|$ so that $\mu(1) = \|\mu\|$ which is enough to imply that μ is positive. The uniqueness of μ follows as in the Remark to Theorem 1.

3. APPLICATIONS OF THE INTEGRAL REPRESENTATION THEOREMS

Application 1 (Bochner's Theorem). Let G be a locally compact abelian group and let \hat{G} denote the (locally compact) character group. Denote