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MISCONCEPTIONS ABOUT K_X

by Steven L. KLEIMAN

There are three common misconceptions about the sheaf K_X of meromorphic functions on a ringed space X: (1) that K_X can be defined as the sheaf associated to the presheaf of total fraction rings,

$$(*) \qquad U \mapsto \Gamma(U, O_X)_{tot},$$

see [EGA IV$_4$, 20.1.3, p. 227] and [1, (3.2), p. 137]; (2) that the stalks $K_{X,x}$ are equal to the total fraction rings $(O_{X,x})_{tot}$, see [EGA IV$_4$, 20.1.1 and 20.1.3, pp. 226-7]; and (3) that if X is a scheme and $U = \operatorname{Spec}(A)$ is an affine open subscheme, then $\Gamma(U, K_X)$ is equal to A_{tot}, or in other words, the presheaf (*) is a sheaf if U ranges exclusively over affines, see [3, Def., p. 140]. These misconceptions will be corrected below with some observations and examples.

The presheaf (*) may fail to exist! Some restriction maps may simply not be defined. For instance, there may be a nonzerodivisor t in $\Gamma(X, O_X)$ whose restriction is a zerodivisor in $\Gamma(U, O_X)$ for some open subset U. Then the fraction $1/t$ in $\Gamma(X, O_X)_{tot}$ has no restriction in $\Gamma(U, O_X)_{tot}$.

For example, let A be a domain with nonzero maximal ideal M. Let P denote the projective line over A, and Y the (closed) fiber over M. Set

$$X = \operatorname{Spec}\left(O_P \oplus O_Y(-1)\right),$$

where $O_Y(-1)$ is viewed as an ideal of square zero. We have

$$\Gamma(X, O_X) = \Gamma(P, O_P) \oplus \Gamma\left(Y, O_Y(-1)\right) = A.$$

Hence any nonzero element t of M is a nonzerodivisor in $\Gamma(X, O_X)$. However, for any affine open subset U of X containing a point of Y, the restriction of t in $\Gamma(U, O_X)$ is zerodivisor. Indeed, $O_Y(-1) \mid U$ is isomorphic to $O_Y \mid U$. So $\Gamma\left(U, O_Y(-1)\right)$ contains a nonzero element s, and obviously $ts = 0$ holds. Note that if A is taken to be a finitely generated algebra over a field, then X is an algebraic scheme for which the presheaf (*) is undefined.

The right way to define K_X is as the sheaf associated to the following presheaf of rings of fractions:

$$(**) \qquad U \mapsto \Gamma(U, O_X)[S(U)^{-1}],$$

where $S(U)$ denotes the set of elements of $\Gamma(U, O_X)$ whose restrictions are nonzerodivisors in the stalks $O_{X,x}$ for all $x \in X$. Note that $S(U)$ is contained in the set of nonzerodivisors in $\Gamma(U, O_X)$. Hence the presheaf (**) is separated; that is, the natural map from it to K_X is injective.

The natural map from O_X to K_X is injective because the one to the presheaf (**) is and the latter is separated (alternatively, and sheaving is exact). Now, let $f: X \to Y$ be a flat map, for example, an open embedding. Then f gives rise naturally to a map, $f^*: K_Y \to f^* K_X$. So K_X will work out well in a theory of (Cartier) divisors.

If X is a scheme and $U = \operatorname{Spec}(A)$ is an affine open subscheme, then $S(U)$ contains (and so consists of) all nonzerodivisors $t \in A$. Indeed, suppose $t(a/b) = 0$ holds in $O_{X,x}$ for some $x \in U$ with $a, b \in A$ and $b(x) \neq 0$. Then $tca = 0$ holds in A for some $c \in A$ with $c(x) \neq 0$. Since t is a nonzerodivisor, $ca = 0$ holds in A. Hence $a/b = 0$ holds in $O_{X,x}$, q.e.d. Therefore, when X is a scheme, the presheaf (*) will be well-defined (and equal to the presheaf (**)) if U ranges exclusively over affines, and the associated sheaf is K_X.

Fix $x \in X$ and set $S_x = \varinjlim \{S(U) \mid U \in x\}$. We have

$$K_{X,x} = S_x^{-1} O_{X,x} \subset (O_{X,x})_{tot}.$$

The inclusion may be proper, even if X is an affine scheme. For example, let B be a domain with a nonzero and nonmaximal ideal p such that p is the intersection of all the maximal ideals M containing it. Set

$$X = \operatorname{Spec}(B \oplus (\oplus_{M \supset p}(B/M))).$$

Let $x \in X$ represent p. Then $K_{X,x}$ is equal to B_p, while $(O_{X,x})_{tot}$ is equal to the fraction field of B.

The presheaf (**) need not be a sheaf. In fact, there is an affine scheme $X = \operatorname{Spec}(A)$ such that A_{tot} is a proper subring of $\Gamma(X, K_X)$. To construct X, fix an algebraically closed ground field k, and a smooth closed cubic E in $\mathbf{P}_k^2$. Let L be a line section of E, and $P \in E$ a k-point such that the divisor $(3P - L)$ has infinite order; for example, take a P whose coordinates are transcendental over the field of definition of E. Let C be a cone in $\mathbf{A}_k^3$ projecting E, and denote by G the generator over P. Take planes H_1 and H_2 through the vertex with no generator in common and neither one containing G. Take a plane H_3 not containing the vertex, parallel to G, but not parallel to any generator on H_1. Denote by U the set of closed points C off $(G \cup H_1) - H_3$. Set

$$X = \operatorname{Spec}\left(O_C \oplus \left(\oplus_{Q \varepsilon U} k\,(Q)\right)\right).$$

Finally, let f be a function on E with a single pole of order 2 at P, and view f as a global section of the sheaf $K_C \oplus \left(\oplus k\,(Q)\right)$, which contains K_X.

Then f is in $\Gamma\,(X, K_X)$ because X is covered by the three affine open subsets $V_i = X - H_i$ and f is easily seen to be in each $\Gamma\,(V_i, O_X)_{tot}$. However, f is not in $\Gamma\,(X, O_X)_{tot}$. Indeed, suppose f is equal to r/s with $r, s \in \Gamma\,(X, O_X)$. Write $s = t + \tau$ with $t \in \Gamma\,(X, O_C)$ and $\tau \in \Gamma\left(X, \oplus k\,(Q)\right)$. Then t is the restriction to C of a polynomial function on $\mathbf{A}_k^3$. So the zero locus $Z\,(t)$ is a hypersurface section of C. Hence, by the choice of $P \in E$, there must be a component Z of $Z\,(t)$ different from G. By construction, U must contain a point Q of Z. Therefore t is a zerodivisor in $\Gamma\,(X, O_X)$, so s is also, q.e.d.

Lastly, consider two common cases: (a) X is a locally noetherian scheme, and (b) X is a reduced scheme whose set of irreducible components is locally finite. In both cases, Assertions (2) and (3) at the beginning are valid, and K_X is given by the formula,

$$K_X = j_*\left(O_X \,|\, \operatorname{Ass}(X)\right),$$

where Ass (X) denotes the set of points $x \in X$ where the maximal ideal of $O_{X,x}$ is associated to 0, and j denotes the inclusion map of Ass (X) into X. These statements are easily verified using the ideas in the proof of [EGA IV_4, 20.2.11, pp. 234-5]. (For a different slant on Case (a), see [4, Lecture 9, 1°, pp. 61-2].)

In Case (b), K_X is quasi-coherent. However, in Case (a) it need not be. For example, let $A = k\,[s, t]$ be the polynomial ring over a field k, and set

$$X = \operatorname{Spec}\left(A \oplus A/(s, t)\right).$$

Though injective, the natural map from $A_{tot}\,[1/s]$ into $A\,[1/s]_{tot}$ is not surjective; the image omits $1/t$. So K_X is not quasi-coherent.

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