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	Kleimen, Steven L
Autor	Kleiman, Sleven L.
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# MISCONCEPTIONS ABOUT $K_X$

# by Steven L. KLEIMAN

There are three common misconceptions about the sheaf  $K_X$  of meromorphic functions on a ringed space X: (1) that  $K_X$  can be defined as the sheaf associated to the presheaf of total fraction rings,

$$(*) \qquad \qquad U \mapsto \Gamma(U, O_X)_{tot},$$

see [EGA IV<sub>4</sub>, 20.1.3, p. 227] and [1, (3.2), p. 137]; (2) that the stalks  $K_{X,x}$  are equal to the total fraction rings  $(O_{X,x})_{tot}$ , see [EGA IV<sub>4</sub>, 20.1.1 and 20.1.3, pp. 226-7]; and (3) that if X is a scheme and U = Spec (A) is an affine open subscheme, then  $\Gamma(U, K_X)$  is equal to  $A_{tot}$ , or in other words, the presheaf (\*) is a sheaf if U ranges exclusively over affines, see [3, Def., p. 140]. These misconceptions will be corrected below with some observations and examples.

The presheaf (\*) may fail to exist! Some restriction maps may simply not be defined. For instance, there may be a nonzerodivisor t in  $\Gamma(X, O_X)$ whose restriction is a zerodivisor in  $\Gamma(U, O_X)$  for some open subset U. Then the fraction 1/t in  $\Gamma(X, O_X)_{tot}$  has no restriction in  $\Gamma(U, O_X)_{tot}$ .

For example, let A be a domain with nonzero maximal ideal M. Let P denote the projective line over A, and Y the (closed) fiber over M. Set

$$X = \operatorname{Spec} \left( O_P \oplus O_Y(-1) \right),$$

where  $O_{Y}(-1)$  is viewed as an ideal of square zero. We have

$$\Gamma(X, O_X) = \Gamma(P, O_P) \oplus \Gamma(Y, O_Y(-1)) = A.$$

Hence any nonzero element t of M is a nonzerodivisor in  $\Gamma(X, O_X)$ . However, for any affine open subset U of X containing a point of Y, the restriction of t in  $\Gamma(U, O_X)$  is zerodivisor. Indeed,  $O_Y(-1) | U$  is isomorphic to  $O_Y | U$ . So  $\Gamma(U, O_Y(-1))$  contains a nonzero element s, and obviously ts = 0 holds. Note that if A is taken to be a finitely generated algebra over a field, then X is an algebraic scheme for which the presheaf (\*) is undefined. The right way to define  $K_X$  is as the sheaf associated to the following presheaf of rings of fractions:

$$(**) \qquad \qquad U \mapsto \Gamma(U, O_X) \left[ S(U)^{-1} \right],$$

where S(U) denotes the set of elements of  $\Gamma(U, O_X)$  whose restrictions are nonzerodivisors in the stalks  $O_{X,x}$  for all  $x \in X$ . Note that S(U) is contained in the set of nonzerodivisors in  $\Gamma(U, O_X)$ . Hence the presheaf (\*\*) is separated; that is, the natural map from it to  $K_X$  is injective.

The natural map from  $O_X$  to  $K_X$  is injective because the one to the presheaf (\*\*) is and the latter is separated (alternatively, and sheaving is exact). Now, let  $f: X \to Y$  be a flat map, for example, an open embedding. Then f gives rise naturally to a map,  $f^*: K_Y \to f^* K_X$ . So  $K_X$  will work out well in a theory of (Cartier) divisors.

If X is a scheme and U = Spec(A) is an affine open subscheme, then S(U) contains (and so consists of) all nonzerodivisors  $t \in A$ . Indeed, suppose t(a/b) = 0 holds in  $O_{X,x}$  for some  $x \in U$  with  $a, b \in A$  and  $b(x) \neq 0$ . Then tca = 0 holds in A for some  $c \in A$  with  $c(x) \neq 0$ . Since t is a nonzerodivisor, ca = 0 holds in A. Hence a/b = 0 holds in  $O_{X,x}$ , q.e.d. Therefore, when X is a scheme, the presheaf (\*) will be well-defined (and equal to the presheaf (\*\*)) if U ranges exclusively over affines, and the associated sheaf is  $K_X$ .

Fix  $x \in X$  and set  $S_x = \lim_{x \to \infty} \{S(U) \mid U \in x\}$ . We have

$$K_{X,x} = S_x^{-1} O_{X,x} \subset (O_{X,x})_{tot}.$$

The inclusion may be proper, even if X is an affine scheme. For example, let B be a domain with a nonzero and nonmaximal ideal p such that p is the intersection of all the maximal ideals M containing it. Set

$$X = \operatorname{Spec} \left( B \oplus \left( \bigoplus_{M \supset p} (B/M) \right) \right).$$

Let  $x \in X$  represent p. Then  $K_{X,x}$  is equal to  $B_p$ , while  $(O_{X,x})_{tot}$  is equal to the fraction field of B.

The presheaf (\*\*) need not be a sheaf. In fact, there is an affine scheme X = Spec (A) such that  $A_{tot}$  is a proper subring of  $\Gamma (X, K_X)$ . To construct X, fix an algebraically closed ground field k, and a smooth closed cubic E in  $\mathbf{P}_k^2$ . Let L be a line section of E, and  $P \in E$  a k-point such that the divisor (3P-L) has infinite order; for example, take a P whose coordinates are transcendental over the field of definition of E. Let C be a cone in  $\mathbf{A}_k^3$  pro-

jecting E, and denote by G the generator over P. Take planes  $H_1$  and  $H_2$  through the vertex with no generator in common and neither one containing G. Take a plane  $H_3$  not containing the vertex, parallel to G, but not parallel to any generator on  $H_1$ . Denote by U the set of closed points C off  $(G \cup H_1) - H_3$ . Set

$$X = \operatorname{Spec} \left( O_{\mathcal{C}} \oplus \left( \oplus_{\mathcal{Q} \in \mathcal{U}} k\left( \mathcal{Q} \right) \right) \right).$$

Finally, let f be a function on E with a single pole of order 2 at P, and view f as a global section of the sheaf  $K_C \oplus (\oplus k(Q))$ , which contains  $K_X$ .

Then f is in  $\Gamma(X, K_X)$  because X is covered by the three affine open subsets  $V_i = X - H_i$  and f is easily seen to be in each  $\Gamma(V_i, O_X)_{tot}$ . However, f is not in  $\Gamma(X, O_X)_{tot}$ . Indeed, suppose f is equal to r/s with  $r, s \in \Gamma(X, O_X)$ . Write  $s = t + \tau$  with  $t \in \Gamma(X, O_C)$  and  $\tau \in \Gamma(X, \bigoplus k(Q))$ . Then t is the restriction to C of a polynomial function on  $A_k^3$ . So the zero locus Z(t) is a hypersurface section of C. Hence, by the choice of  $P \in E$ , there must be a component Z of Z(t) different from G. By construction, U must contain a point Q of Z. Therefore t is a zerodivisor in  $\Gamma(X, O_X)$ , so s is also, q.e.d.

Lastly, consider two common cases: (a) X is a locally noetherian scheme, and (b) X is a reduced scheme whose set of irreducible components is locally finite. In both cases, Assertions (2) and (3) at the beginning are valid, and  $K_X$  is given by the formula,

$$K_X = j_* \left( O_X \, | \, \operatorname{Ass} \left( X \right) \right),$$

where Ass (X) denotes the set of points  $x \in X$  where the maximal ideal of  $O_{X,x}$  is associated to 0, and j denotes the inclusion map of Ass (X) into X. These statements are easily verified using the ideas in the proof of [EGA IV<sub>4</sub>, 20.2.11, pp. 234-5]. (For a different slant on Case (a), see [4, Lecture 9, 1°, pp. 61-2].)

In Case (b),  $K_X$  is quasi-coherent. However, in Case (a) it need not be. For example, let A = k [s, t] be the polynomial ring over a field k, and set

$$X = \operatorname{Spec} \left( A \oplus A / (s, t) \right).$$

Though injective, the natural map from  $A_{tot}$  [1/s] into A [1/s]<sub>tot</sub> is not surjective; the image omits 1/t. So  $K_X$  is not quasi-coherent.

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Steven L. Kleiman

Department of Mathematics M.I.T., 2-278 Cambridge, Mass. 02139