

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 25 (1979)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: MISCONCEPTIONS ABOUT $\$K_x\$$
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DOI: <https://doi.org/10.5169/seals-50379>

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MISCONCEPTIONS ABOUT K_X

by Steven L. KLEIMAN

There are three common misconceptions about the sheaf K_X of meromorphic functions on a ringed space X : (1) that K_X can be defined as the sheaf associated to the presheaf of total fraction rings,

$$(*) \quad U \mapsto \Gamma(U, \mathcal{O}_X)_{tot},$$

see [EGA IV₄, 20.1.3, p. 227] and [1, (3.2), p. 137]; (2) that the stalks $K_{X,x}$ are equal to the total fraction rings $(\mathcal{O}_{X,x})_{tot}$, see [EGA IV₄, 20.1.1 and 20.1.3, pp. 226-7]; and (3) that if X is a scheme and $U = \text{Spec}(A)$ is an affine open subscheme, then $\Gamma(U, K_X)$ is equal to A_{tot} , or in other words, the presheaf (*) is a sheaf if U ranges exclusively over affines, see [3, Def., p. 140]. These misconceptions will be corrected below with some observations and examples.

The presheaf (*) may fail to exist! Some restriction maps may simply not be defined. For instance, there may be a nonzerodivisor t in $\Gamma(X, \mathcal{O}_X)$ whose restriction is a zerodivisor in $\Gamma(U, \mathcal{O}_X)$ for some open subset U . Then the fraction $1/t$ in $\Gamma(X, \mathcal{O}_X)_{tot}$ has no restriction in $\Gamma(U, \mathcal{O}_X)_{tot}$.

For example, let A be a domain with nonzero maximal ideal M . Let P denote the projective line over A , and Y the (closed) fiber over M . Set

$$X = \text{Spec}(\mathcal{O}_P \oplus \mathcal{O}_Y(-1)),$$

where $\mathcal{O}_Y(-1)$ is viewed as an ideal of square zero. We have

$$\Gamma(X, \mathcal{O}_X) = \Gamma(P, \mathcal{O}_P) \oplus \Gamma(Y, \mathcal{O}_Y(-1)) = A.$$

Hence any nonzero element t of M is a nonzerodivisor in $\Gamma(X, \mathcal{O}_X)$. However, for any affine open subset U of X containing a point of Y , the restriction of t in $\Gamma(U, \mathcal{O}_X)$ is zerodivisor. Indeed, $\mathcal{O}_Y(-1)|_U$ is isomorphic to $\mathcal{O}_Y|_U$. So $\Gamma(U, \mathcal{O}_Y(-1))$ contains a nonzero element s , and obviously $ts = 0$ holds. Note that if A is taken to be a finitely generated algebra over a field, then X is an algebraic scheme for which the presheaf (*) is undefined.

The right way to define K_X is as the sheaf associated to the following presheaf of rings of fractions:

$$(**) \quad U \mapsto \Gamma(U, \mathcal{O}_X) [S(U)^{-1}],$$

where $S(U)$ denotes the set of elements of $\Gamma(U, \mathcal{O}_X)$ whose restrictions are nonzerodivisors in the stalks $\mathcal{O}_{X,x}$ for all $x \in U$. Note that $S(U)$ is contained in the set of nonzerodivisors in $\Gamma(U, \mathcal{O}_X)$. Hence the presheaf $(**)$ is separated; that is, the natural map from it to K_X is injective.

The natural map from \mathcal{O}_X to K_X is injective because the one to the presheaf $(**)$ is and the latter is separated (alternatively, and sheaving is exact). Now, let $f: X \rightarrow Y$ be a flat map, for example, an open embedding. Then f gives rise naturally to a map, $f^*: K_Y \rightarrow f^* K_X$. So K_X will work out well in a theory of (Cartier) divisors.

If X is a scheme and $U = \text{Spec}(A)$ is an affine open subscheme, then $S(U)$ contains (and so consists of) all nonzerodivisors $t \in A$. Indeed, suppose $t(a/b) = 0$ holds in $\mathcal{O}_{X,x}$ for some $x \in U$ with $a, b \in A$ and $b(x) \neq 0$. Then $tca = 0$ holds in A for some $c \in A$ with $c(x) \neq 0$. Since t is a nonzerodivisor, $ca = 0$ holds in A . Hence $a/b = 0$ holds in $\mathcal{O}_{X,x}$, q.e.d. Therefore, when X is a scheme, the presheaf $(*)$ will be well-defined (and equal to the presheaf $(**)$) if U ranges exclusively over affines, and the associated sheaf is K_X .

Fix $x \in X$ and set $S_x = \varinjlim \{S(U) \mid U \in x\}$. We have

$$K_{X,x} = S_x^{-1} \mathcal{O}_{X,x} \subset (\mathcal{O}_{X,x})_{\text{tot}}.$$

The inclusion may be proper, even if X is an affine scheme. For example, let B be a domain with a nonzero and nonmaximal ideal p such that p is the intersection of all the maximal ideals M containing it. Set

$$X = \text{Spec}(B \oplus (\oplus_{M \supset p} (B/M))).$$

Let $x \in X$ represent p . Then $K_{X,x}$ is equal to B_p , while $(\mathcal{O}_{X,x})_{\text{tot}}$ is equal to the fraction field of B .

The presheaf $(**)$ need not be a sheaf. In fact, there is an affine scheme $X = \text{Spec}(A)$ such that A_{tot} is a proper subring of $\Gamma(X, K_X)$. To construct X , fix an algebraically closed ground field k , and a smooth closed cubic E in \mathbf{P}_k^2 . Let L be a line section of E , and $P \in E$ a k -point such that the divisor $(3P - L)$ has infinite order; for example, take a P whose coordinates are transcendental over the field of definition of E . Let C be a cone in \mathbf{A}_k^3 pro-

jecting E , and denote by G the generator over P . Take planes H_1 and H_2 through the vertex with no generator in common and neither one containing G . Take a plane H_3 not containing the vertex, parallel to G , but not parallel to any generator on H_1 . Denote by U the set of closed points C off $(G \cup H_1) - H_3$. Set

$$X = \text{Spec} (O_C \oplus (\oplus_{Q \in U} k(Q))).$$

Finally, let f be a function on E with a single pole of order 2 at P , and view f as a global section of the sheaf $K_C \oplus (\oplus k(Q))$, which contains K_X .

Then f is in $\Gamma(X, K_X)$ because X is covered by the three affine open subsets $V_i = X - H_i$ and f is easily seen to be in each $\Gamma(V_i, O_X)_{tot}$. However, f is not in $\Gamma(X, O_X)_{tot}$. Indeed, suppose f is equal to r/s with $r, s \in \Gamma(X, O_X)$. Write $s = t + \tau$ with $t \in \Gamma(X, O_C)$ and $\tau \in \Gamma(X, \oplus k(Q))$. Then t is the restriction to C of a polynomial function on A_k^3 . So the zero locus $Z(t)$ is a hypersurface section of C . Hence, by the choice of $P \in E$, there must be a component Z of $Z(t)$ different from G . By construction, U must contain a point Q of Z . Therefore t is a zerodivisor in $\Gamma(X, O_X)$, so s is also, q.e.d.

Lastly, consider two common cases: (a) X is a locally noetherian scheme, and (b) X is a reduced scheme whose set of irreducible components is locally finite. In both cases, Assertions (2) and (3) at the beginning are valid, and K_X is given by the formula,

$$K_X = j_* (O_X | \text{Ass}(X)),$$

where $\text{Ass}(X)$ denotes the set of points $x \in X$ where the maximal ideal of $O_{X,x}$ is associated to 0, and j denotes the inclusion map of $\text{Ass}(X)$ into X . These statements are easily verified using the ideas in the proof of [EGA IV₄, 20.2.11, pp. 234-5]. (For a different slant on Case (a), see [4, Lecture 9, 1°, pp. 61-2].)

In Case (b), K_X is quasi-coherent. However, in Case (a) it need not be. For example, let $A = k[s, t]$ be the polynomial ring over a field k , and set

$$X = \text{Spec} (A \oplus A/(s, t)).$$

Though injective, the natural map from $A_{tot}[1/s]$ into $A[1/s]_{tot}$ is not surjective; the image omits $1/t$. So K_X is not quasi-coherent.

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(Reçu le 30 october 1978)

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