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Also Schwarz's inequality yields

$$\int_{R_0+1}^{R_n} h_2(t,\phi)\frac{dt}{t}\int_{R_0+1}^{R_n}\frac{dt}{t\,h_2(t,\phi)} \geqslant \left\{\log\left(\frac{R_n}{R_0+1}\right)\right\}^2,$$

i.e.

$$\int_{R_0+1}^{R_n}\frac{dt}{t\,h_2(t,\phi)} \geqslant \left\{\log\frac{R_n}{R_0+1}\right\}^2 \Big/ \int_{R_0+1}^{R_n} h_2(t,\phi)\frac{dt}{t}$$

$$> \frac{2\mu+2\delta}{\pi}\log R_n$$

for all large n, where δ is a positive constant, in view of (13). Thus (15) yields

$$\omega_n(z) = O(R_n^{-\mu-\delta}), \quad \text{as } n \to \infty . \tag{16}$$

Also since $u(z) \leqslant (K+1)B(R_n)$ on $|z| = R_n$, we deduce finally that

$$u(z) \leqslant (K+1)B(R_n)\,\omega_n(z)$$

in Δ_n and now (12) and (16) yield (14) for any point in Δ. In particular for z on S, we deduce (11) as required. This proves the Lemma.

4. Conclusions

It is not difficult to obtain a contradiction from the above Lemma. We may assume without loss of generality that the angle is given by $S: |\arg z| < \frac{\pi}{2\mu}$. Since $f(z)$ is bounded in S, we deduce that $\log|f(z)|$ is bounded above in S by the Poisson integral of the boundary values on the arms $\arg z = \mp\pi/(2\mu)$. This leads, for $K > 1$, to

$$\log|f(re^{i\theta})| < -A(\mu)(K-1)r^{\mu}\int_r^{\infty}\frac{B(t)\,dt}{t^{\mu+1}}, \quad |\theta| < \frac{\pi}{2\mu}, \tag{17}$$
$$0 < r < \infty ,$$

where the constant $A(\mu)$ depends only on μ.

Given any constant $C > 1$, we can, since f has lower order μ find a sequence r_n tending to infinity with n and such that

$$B(t) > \frac{1}{2}\left(\frac{t}{r_n}\right)^{\mu} B(r_n), \quad r_n \leqslant t \leqslant Cr_n$$

Now (17) yields

$$\log |f(r_n e^{i\theta})| < -A(\mu)(K-1)B(r_n)\frac{1}{2}\int_{r_n}^{Cr_n}\frac{dt}{t}$$
$$= -\frac{1}{2}A(\mu)(K-1)B(r_n)\log C.$$

Thus

$$\int_{-\pi}^{\pi}\log |f(r_n e^{i\theta})|\, d\theta \leqslant -\frac{\pi}{\mu}A(\mu)(K-1)B(r_n)\log C$$
$$+\left(2\pi-\frac{\pi}{\mu}\right)B(r_n).$$

This contradicts Jensen's formula if C is sufficiently large, since the left hand side is bounded below.

We can also obtain some conclusions if $K = 1$. In this case we note that if $S: \alpha_1 \leqslant \arg z \leqslant \alpha_2$ is the angle constructed in Lemma 1 then, since Γ lies almost entirely in S, we deduce for large r that

$$\inf_{\alpha_1 \leqq \theta \leqq \alpha_2} \log |f(re^{i\theta})| \leqslant -\frac{1}{3}B(r).$$

Since f is bounded above in S it follows from an earlier Theorem of mine [5] that

$$\overline{\lim_{r\to\infty}}\; B(r)\, r^{\pi/(\alpha_2-\alpha_1)} < \infty.$$

Thus $B(r)$ has order $\lambda = \mu$ and f cannot have maximal type. Further $\alpha_2 - \alpha_1 = \pi/\lambda$ and from this we deduce that as $z = re^{i\theta} \to \infty$ on Γ outside a set of r of logarithmic density zero

$$\theta = arg\ z \to \frac{1}{2}(\alpha_1+\alpha_2),$$

so that Γ has a preferred direction. If $\mu = \infty$, we must have $\alpha_1 = \alpha_2$, so that Γ has a unique limiting direction.

We also note that $\mu > 1$, unless $f(z) \equiv e^{(az+b)}$. For we have seen that f cannot have order 1, maximal type. However if $f(z) \not\equiv e^{az+b}$ and f has order one mean type, or minimal type then an earlier theorem of mine [6] shows that $\mu(r)\, M(r)$ cannot be bounded.

Finally let me say a few words concerning the case of infinite order. In this case we assume $K > 1$ and define

$$\mu(r) = \inf \frac{\log B(r_2) - \log B(r_1) + A_1(K)}{\log r_2 - \log r_1}$$

where the inf is taken over all pairs r_1, r_2, such that $r < r_1 < r_2 < \infty$, and

$$A_1(K) = \log \left\{ \frac{20(1+K)}{K-1} \right\}.$$

The quantity $\mu(r)$ plays a similar role to the lower order μ in the above argument and

$$\log |f(z)| < -\frac{K-1}{4} B(z), \ \alpha_1(r) < arg\ z < \alpha_2(r), \ |z| < r,$$

where $\alpha_2 - \alpha_1 \geqslant \pi/\mu(r)$. From this and the fact that $\mu(r)$ increases with r it is possible to obtain a contradiction.

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