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Artikel: HOW QUICKLY CAN AN ENTIRE FUNCTION TEND TO ZERO ALONG A CURVE ?

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Also Schwarz's inequality yields

$$\int_{R_0+1}^{R_n} h_2(t,\phi)\frac{dt}{t}\int_{R_0+1}^{R_n}\frac{dt}{t\,h_2(t,\phi)} \geqslant \left\{\log\left(\frac{R_n}{R_0+1}\right)\right\}^2,$$

i.e.

$$\int_{R_0+1}^{R_n}\frac{dt}{t\,h_2(t,\phi)} \geqslant \left\{\log\frac{R_n}{R_0+1}\right\}^2 \Big/ \int_{R_0+1}^{R_n} h_2(t,\phi)\frac{dt}{t}$$

$$> \frac{2\mu+2\delta}{\pi}\log R_n$$

for all large n, where δ is a positive constant, in view of (13). Thus (15) yields

$$\omega_n(z) = O(R_n^{-\mu-\delta}), \quad \text{as } n\to\infty\,. \tag{16}$$

Also since $u(z) \leqslant (K+1)\,B(R_n)$ on $|z| = R_n$, we deduce finally that

$$u(z) \leqslant (K+1)\,B(R_n)\,\omega_n(z)$$

in Δ_n and now (12) and (16) yield (14) for any point in Δ. In particular for z on S, we deduce (11) as required. This proves the Lemma.

4. CONCLUSIONS

It is not difficult to obtain a contradiction from the above Lemma. We may assume without loss of generality that the angle is given by $S: |\arg z| < \frac{\pi}{2\mu}$. Since $f(z)$ is bounded in S, we deduce that $\log|f(z)|$ is bounded above in S by the Poisson integral of the boundary values on the arms $\arg z = \mp\,\pi/(2\mu)$. This leads, for $K>1$, to

$$\log|f(re^{i\theta})| < -A(\mu)(K-1)\,r^{\mu}\int_r^{\infty}\frac{B(t)\,dt}{t^{\mu+1}}, \quad |\theta| < \frac{\pi}{2\mu}, \tag{17}$$

$$0 < r < \infty\,,$$

where the constant $A(\mu)$ depends only on μ.

Given any constant $C>1$, we can, since f has lower order μ find a sequence r_n tending to infinity with n and such that

$$B(t) > \frac{1}{2}\left(\frac{t}{r_n}\right)^{\mu} B(r_n), \quad r_n \leqslant t \leqslant Cr_n$$

Now (17) yields

$$\log |f(r_n e^{i\theta})| < -A(\mu)(K-1)B(r_n)\frac{1}{2}\int_{r_n}^{Cr_n}\frac{dt}{t}$$
$$= -\frac{1}{2}A(\mu)(K-1)B(r_n)\log C .$$

Thus

$$\int_{-\pi}^{\pi}\log |f(r_n e^{i\theta})|\,d\theta \leqslant -\frac{\pi}{\mu}A(\mu)(K-1)B(r_n)\log C$$
$$+\left(2\pi-\frac{\pi}{\mu}\right)B(r_n) .$$

This contradicts Jensen's formula if C is sufficiently large, since the left hand side is bounded below.

We can also obtain some conclusions if $K = 1$. In this case we note that if $S: \alpha_1 \leqslant \arg z \leqslant \alpha_2$ is the angle constructed in Lemma 1 then, since Γ lies almost entirely in S, we deduce for large r that

$$\inf_{\alpha_1 \leqq \theta \leqq \alpha_2}\log |f(re^{i\theta})| \leqslant -\frac{1}{3}B(r) .$$

Since f is bounded above in S it follows from an earlier Theorem of mine [5] that

$$\overline{\lim_{r\to\infty}}\; B(r)\, r^{\pi/(\alpha_2-\alpha_1)} < \infty .$$

Thus $B(r)$ has order $\lambda = \mu$ and f cannot have maximal type. Further $\alpha_2 - \alpha_1 = \pi/\lambda$ and from this we deduce that as $z = re^{i\theta} \to \infty$ on Γ outside a set of r of logarithmic density zero

$$\theta = arg\ z \to \frac{1}{2}(\alpha_1+\alpha_2) ,$$

so that Γ has a preferred direction. If $\mu = \infty$, we must have $\alpha_1 = \alpha_2$, so that Γ has a unique limiting direction.

We also note that $\mu > 1$, unless $f(z) \equiv e^{(az+b)}$. For we have seen that f cannot have order 1, maximal type. However if $f(z) \not\equiv e^{az+b}$ and f has order one mean type, or minimal type then an earlier theorem of mine [6] shows that $\mu(r)\,M(r)$ cannot be bounded.

Finally let me say a few words concerning the case of infinite order. In this case we assume $K > 1$ and define

$$\mu(r) = \inf\frac{\log B(r_2) - \log B(r_1) + A_1(K)}{\log r_2 - \log r_1}$$

where the inf is taken over all pairs r_1, r_2, such that $r < r_1 < r_2 < \infty$, and

$$A_1(K) = \log \left\{ \frac{20(1+K)}{K-1} \right\}.$$

The quantity $\mu(r)$ plays a similar role to the lower order μ in the above argument and

$$\log |f(z)| < -\frac{K-1}{4} B(z), \ \alpha_1(r) < arg\ z < \alpha_2(r), \ |z| < r,$$

where $\alpha_2 - \alpha_1 \geqslant \pi/\mu(r)$. From this and the fact that $\mu(r)$ increases with r it is possible to obtain a contradiction.

REFERENCES

[1] Beurling, A. Some theorems on boundedness of analytic functions. *Duke Math. J. 16* (1949), pp. 355-359.

[2] Hadamard, J. Etude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *J. de Math. (4) 9* (1893), pp. 171-215.

[3] Haymann, W. K. Remarks on Ahlfors' distortion theorem. *Quart. J. Math., Oxford Ser. 19* (1948), pp. 33-53.

[4] —— The minimum modulus of large integral functions. *Proc. London Math. Soc. (3) 2* (1952), pp. 469-512.

[5] —— Questions of regularity connected with the Phragmén-Lindelöf principle. *J. Math. Pures Appl. (9) 35* (1956), pp. 115-126.

[6] —— The minimum modulus of integral functions of order one. *J. Analyse Math. 28* (1975), pp. 171-212.

[7] Kjellberg, B. A theorem on the minimum modulus of entire functions. *Math. Scand. 12* (1963), pp. 5-11.

[8] Littlewood, J. E. A general theorem on integral functions of finite order. *Proc. London Math. Soc. (2) 6* (1908), pp. 189-204.

[9] Valiron, G. Sur les fonctions entières d'ordre nul et d'ordre fini, et en particulier les fonctions à correspondance régulière. *Ann. Fac. Sci. Univ. Toulouse (3) 5* (1913), pp. 117-257.

[10] Wiman, A. Über eine Eigenschaft der ganzen Funktionen von der Höhe null. *Math. Annalen 76* (1915), pp. 197-211.

[11] —— Über den Zusammenhang zwischen dem Maximalbetrage einer analytischen Funktion und dem grössten Betrage bei gegebenem Argumente der Funktion. *Acta Math. 41* (1918), pp. 1-28.

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