for some arbitrarily large $z = re^{i\theta}$ on $E$. Here $A_0$ is an absolute but presumably very large constant. I had conjectured that the result holds for any $A_0 > 1$. Soon afterwards Beurling showed Kjellberg in a conversation that (8) holds for any $A_0 > 3$. Beurling's argument is as follows.
We write

$$B(r) = \log^+ M(r) = \max\{0, \log M(r)\}, \quad B(z) = B(|z|),$$

and suppose that for some $K \geqslant 1$, we have

$$\log |f(z)| < -KB(z), \tag{9}$$

on a Jordan curve $\Gamma$ joining $z = 0$, $z_0 = Re^{i\theta}$. Then we deduce that

$$\log |f(re^{i\theta})| \leqslant -\frac{K-1}{2} B(r), \quad 0 < r < R. \tag{10}$$

To see this we suppose that $S$: $[r_1, r_2]$ is a maximal interval such that $re^{i\theta}$ does not lie on $\Gamma$, for $r_1 < r < r_2$. Let $\gamma$ be the arc of $\Gamma$ with end points $r_1 e^{i\theta}$, $r_2 e^{i\theta}$, let $D$ be the domain bounded by $\gamma$ and $S$, $D^*$ the reflexion of $D$ in $S$ and $\Delta = D \cup S \cup D^*$. In $\Delta$ we consider the function

$$u(z) = \log |f(z)| + \log |f(z^*)| + (K-1) B(z)$$

where $z^*$ is the reflexion of $z$ in $S$. Clearly $u(z)$ is subharmonic in $\Delta$ and, for $z$ on the boundary of $\Delta$, either $z$ or $z^*$ lies on $\Gamma$. Thus

$$u(z) \leqslant 0$$

in $\Delta$ and in particular on $S$. We deduce that

$$2 \log |f(re^{i\theta})| \leqslant -(K-1) B(r), \quad r_1 < r < r_2$$

and this yields (10). Hence if $K > 3$, we deduce that $f$ is constant from Beurling's theorem.

Recalling his earlier conversation with Beurling, Kjellberg went on to prove 18 months ago that (8) holds for any $A_0 > 1$ at least when $f$ has finite order and I managed to extend the result to the case of infinite order. Our joint paper will be published in the Turan memorial volume. I should like to describe briefly the idea behind this proof.

## 3. An extended reflexion principle

Let us return to the above reflexion argument. We assume now that (9) holds on some curve $\Gamma$ going from 0 to $\infty$, where $K \geqslant 1$. Then the reflexion principle shows that

$$\log |f(z)| \leqslant -\frac{K-1}{2} B(z) \tag{11}$$

on any ray joining the origin to some point on $\Gamma$. Kjellberg extended this to prove the following

Lemma. *If $f$ has lower order $\mu < \infty$. Then (11) holds in some sector of opening at least $\pi/\mu$.*

From this he was able to obtain a contradiction if $K > 1$. To prove the Lemma we let $\theta_1$, $\theta_2$ be the lower and upper limits of $\arg z$ as $z \to \infty$ on $\Gamma$. Then the above argument shows that (11) holds for $\theta_1 < \arg z < \theta_2$. Thus is $\theta_2 - \theta_1 \geqslant \pi/\mu$, the Lemma is proved.

Suppose now that $\theta_2 - \theta_1 < \pi/\mu$. We may assume that $\mu \geqslant 1$, since otherwise our conclusion follows from (5) in which $\lambda$ can be replaced by $\mu$ according to a Theorem of Kjellberg [7]. We choose a sequence $R_n$ which tends to $\infty$ with $n$ and is such that

$$\log B(R_n) < (\mu + o(1)) \log R_n . \tag{12}$$

We now define quantities $\alpha_1$, $\alpha_2$ as follows. For any fixed $\phi_1 < \theta_1$ and sufficiently large $R$, we define $h_1(R, \phi_1)$ to be the largest number such that the arc

$$\phi_1 < \arg z < \phi_1 + h_1(R, \phi_1), \quad |z| = R$$

does not meet $\Gamma$. Clearly $h_1 \leqslant \theta_2 + o(1) - \phi_1$ for large $R$. Similarly, for $\phi_2 > \theta_2$, we define $h_2(R, \phi_2)$ to be the largest number such that the arc

$$\phi_2 - h_2(R, \phi_2) < \arg z < \phi_2, \quad |z| = R$$

does not meet $\Gamma$. Then $\alpha$ is defined to be the greatest lower bound of all $\phi_1 < \theta_1$ such that, for a fixed large $R_0$, we have

$$\varliminf_{n\to\infty} \frac{1}{\log R_n} \int_{R_0}^{R_n} h_1(t, \phi_1) \frac{dt}{t} < \frac{\pi}{2\mu} .$$

If there are no such numbers $\phi_1$, we define $\alpha_1 = \theta_1$. Also $\alpha_2$ is defined similarly as the least upper bound of all $\phi_2 > \theta_2$ such that

$$\varliminf_{n\to\infty} \frac{1}{\log R_n} \int_{R_0}^{R_n} h_2(t, \phi_2) \frac{dt}{t} < \frac{\pi}{2\mu} . \tag{13}$$

If there are no such $\phi_2$ we define $\alpha_2 = \theta_2$.

Suppose now that $\phi_1 < \alpha_1$, $\phi_2 > \alpha_2$. Then we deduce that for a fixed large $R_0$ and all sufficiently large $n$

$$(\phi_2 - \phi_1) \log \frac{R_n}{R_0} = \int_{R_0}^{R_n} (\phi_2 - \phi_1) \frac{dt}{t} \geqslant \int_{R_0}^{R_n} \{h_1(t, \phi_1) + h_2(t, \phi_2)\} \frac{dt}{t}$$

$$\geqslant \left(\frac{\pi}{\mu} + o(1)\right) \log R_n .$$

Thus $\phi_2 - \phi_1 \geqslant \pi/\mu$, and hence $\alpha_2 - \alpha_1 \geqslant \pi/\mu$.
On the other hand we can show that (11) holds for $0 < |z| < \infty$, $\alpha_1 \leqslant \arg z \leqslant \alpha_2$.

To see this we choose $\phi$, such that $\alpha_1 < \phi < \alpha_2$ and assume that $\Gamma$ does not meet the ray $\arg z = \phi$ for arbitrarily large $z$, since otherwise the conclusion follows from (10). In particular (11) holds for $\theta_1 < \phi < \theta_2$ and hence by continuity also for $\phi = \theta_1$ or $\theta_2$. Thus we may assume that either $\alpha_1 < \phi < \theta_1$ or $\theta_2 < \phi < \alpha_2$. Suppose e.g. that the latter inequality holds, so that in particular $\alpha_2 > \theta_2$. Let $z_0 = Re^{i\phi}$ be the last intersection of $\arg z = \phi$ with $\Gamma$. Let $D$ be the domain bounded by the arc $\Gamma_0$ of $\Gamma$ from $z_0$ to $\infty$ and by the segment $S: z = te^{i\phi}$, $R_0 < t < \infty$.
Let $D^*$ be the reflexion of $D$ in $S$ and set $\Delta = D \cup S \cup D^*$.
We consider

$$u(z) = \log |f(z)| + \log |f(z^*)| + (K-1) B(z)$$

in $\Delta$, where $z^*$ denotes the reflexion of $z$ in $S$, and proceed to show that

$$u(z) \leqslant 0 \text{ in } \Delta . \tag{14}$$

By our construction (14) holds on the finite boundary $\Gamma_0 \cup \Gamma_0^*$ of $\Delta$.
To deal with points at $\infty$ we combine (12) and (13).
We choose a large $n$ and define $\omega_n(z)$ to be the harmonic measure of the circle $|z| = R_n$, with respect to the subdomain $\Delta_n$ of $\Delta$ bounded by $|z| = R_n$, $\Gamma_0$, $\Gamma_0^*$ and containing the part $R_0 < t < R_n$ of the segment $S$. If $z$ is a fixed point of and we let $n$ tend to $\infty$, then standard estimates yield [1])

$$\omega_n(z) \leqslant \exp\left\{-\pi \int_{R_0+1}^{R_n} \frac{dt}{2 h_2(t, \phi)} + O(1)\right\}, \quad \text{as } n \to \infty . \tag{15}$$

[1]) We may map $\Delta_n$ onto a half strip and then apply Ahlfors' distortion theorem in the form given in [3].

Also Schwarz's inequality yields

$$\int_{R_0+1}^{R_n} h_2(t,\phi)\,\frac{d\,t}{t}\int_{R_0+1}^{R_n}\frac{d\,t}{t\,h_2(t,\phi)} \geqslant \left\{\log\left(\frac{R_n}{R_0+1}\right)\right\}^2 ,$$

i.e.

$$\int_{R_0+1}^{R_n}\frac{d\,t}{t\,h_2(t,\phi)} \geqslant \left\{\log\frac{R_n}{R_0+1}\right\}^2 \Big/ \int_{R_0+1}^{R_n} h_2(t,\phi)\,\frac{d\,t}{t}$$

$$> \frac{2\,\mu+2\,\delta}{\pi}\log\,R_n$$

for all large $n$, where $\delta$ is a positive constant, in view of (13). Thus (15) yields

$$\omega_n(z) = O(R_n^{-\mu-\delta}), \quad \text{as } n\to\infty\,. \tag{16}$$

Also since $u(z)\leqslant(K+1)\,B(R_n)$ on $|z| = R_n$, we deduce finally that

$$u(z)\leqslant(K+1)\,B(R_n)\,\omega_n(z)$$

in $\Delta_n$ and now (12) and (16) yield (14) for any point in $\Delta$. In particular for $z$ on $S$, we deduce (11) as required. This proves the Lemma.

## 4. Conclusions

It is not difficult to obtain a contradiction from the above Lemma. We may assume without loss of generality that the angle is given by $S: |\arg z| < \dfrac{\pi}{2\,\mu}$. Since $f(z)$ is bounded in $S$, we deduce that $\log|f(z)|$ is bounded above in $S$ by the Poisson integral of the boundary values on the arms $\arg z = \mp\,\pi/(2\mu)$. This leads, for $K>1$, to

$$\log\,|f(re^{i\theta})| < -\,A(\mu)(K-1)\,r^{\mu}\int_r^{\infty}\frac{B(t)\,d\,t}{t^{\mu+1}}, \quad |\theta| < \frac{\pi}{2\,\mu}, \quad 0<r<\infty\,, \tag{17}$$

where the constant $A(\mu)$ depends only on $\mu$.

Given any constant $C>1$, we can, since $f$ has lower order $\mu$ find a sequence $r_n$ tending to infinity with $n$ and such that

$$B(t) > \frac{1}{2}\left(\frac{t}{r_n}\right)^{\mu} B(r_n)\,, \quad r_n\leqslant t\leqslant Cr_n$$