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Artikel: HOW QUICKLY CAN AN ENTIRE FUNCTION TEND TO ZERO ALONG A CURVE ?

Autor: Hayman, W. K.

Kapitel: 2. The case when E is a curve

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for a sequence of r tending to ∞. This had been conjectured by Littlewood [8] who proved the corresponding theorem with $\cos(2\pi\lambda)$ instead of $\cos(\pi\lambda)$. The result is valid for $0 \leqslant \lambda \leqslant 1$.

If $1 < \lambda < \infty$, Littlewood [8] also proved that there exists a positive constant $C(\lambda)$ such that

$$\mu(r) > M(r)^{-C(\lambda)-\varepsilon}$$

for a sequence of r tending to ∞. However the correct value of $C(\lambda)$ is unknown for $\lambda > 1$. It turns out that the formula (1) with exponential factors is much harder to work with than (2). Wiman [11] conjectured that $C(\lambda) = 1$ for $\lambda > 1$, a result which is true if $f(z)$ has no zeros. Later Beurling [1] proved a corresponding theorem for the case when $f(z)$ attains its minimum on a ray. Nevertheless Wiman's conjecture is false and the correct order of magnitude of Littlewood's constant $C(\lambda)$ is $\log \lambda$ as $\lambda \to \infty$. For infinite order the corresponding Theorem is [4].

$$\mu(r) > M(r)^{-A\log\log\log M(r)}, \tag{6}$$

where the best value of A lies between .09 and 11.03.

Since the theory of $\mu(r)$ is thus rather unsatisfactory for $\lambda > 1$ it is natural to consider other cases of E. Suppose first that E is a ray $\arg z = \theta$ and that $K > 1$. Then Beurling [1] showed that if

$$|f(re^{i\theta})| < M(r)^{-K}, \tag{7}$$

for $0 < r < R$, we have

$$|f(z)| < 1, \quad |z| = C_1(K)R,$$

where the constant $C_1(K)$ depends only on K. If R can be chosen arbitrarily large, we deduce at once that $f(z)$ is bounded on a sequence of large circles $|z| = C_1 R$, so that f is constant by Liouville's theorem. Thus for non-constant f (7) cannot be true for all r (or all large r) and a fixed θ.

2. The case when E is a curve

It is natural to consider the case when E is an unbounded connected set or equivalently a curve going to ∞ and this is the topic I mainly wish to discuss today. By a rather involved method I had shown [4] that in this case

$$|f(z)| > M(r)^{-A_0}, \tag{8}$$

for some arbitrarily large $z = re^{i\theta}$ on E. Here A_0 is an absolute but presumably very large constant. I had conjectured that the result holds for any $A_0 > 1$. Soon afterwards Beurling showed Kjellberg in a conversation that (8) holds for any $A_0 > 3$. Beurling's argument is as follows.
We write

$$B(r) = \log^+ M(r) = \max\{0, \log M(r)\}, \quad B(z) = B(|z|),$$

and suppose that for some $K \geqslant 1$, we have

$$\log |f(z)| < -KB(z), \tag{9}$$

on a Jordan curve Γ joining $z = 0$, $z_0 = Re^{i\theta}$. Then we deduce that

$$\log |f(re^{i\theta})| \leqslant -\frac{K-1}{2} B(r), \quad 0 < r < R. \tag{10}$$

To see this we suppose that S: $[r_1, r_2]$ is a maximal interval such that $re^{i\theta}$ does not lie on Γ, for $r_1 < r < r_2$. Let γ be the arc of Γ with end points $r_1 e^{i\theta}$, $r_2 e^{i\theta}$, let D be the domain bounded by γ and S, D^* the reflexion of D in S and $\Delta = D \cup S \cup D^*$. In Δ we consider the function

$$u(z) = \log |f(z)| + \log |f(z^*)| + (K-1) B(z)$$

where z^* is the reflexion of z in S. Clearly $u(z)$ is subharmonic in Δ and, for z on the boundary of Δ, either z or z^* lies on Γ. Thus

$$u(z) \leqslant 0$$

in Δ and in particular on S. We deduce that

$$2 \log |f(re^{i\theta})| \leqslant -(K-1) B(r), \quad r_1 < r < r_2$$

and this yields (10). Hence if $K > 3$, we deduce that f is constant from Beurling's theorem.

Recalling his earlier conversation with Beurling, Kjellberg went on to prove 18 months ago that (8) holds for any $A_0 > 1$ at least when f has finite order and I managed to extend the result to the case of infinite order. Our joint paper will be published in the Turan memorial volume. I should like to describe briefly the idea behind this proof.

3. An extended reflexion principle

Let us return to the above reflexion argument. We assume now that (9) holds on some curve Γ going from 0 to ∞, where $K \geqslant 1$. Then the reflexion principle shows that