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HOW QUICKLY CAN AN ENTIRE FUNCTION TEND TO ZERO ALONG A CURVE ? [1]

by W. K. HAYMAN

1. INTRODUCTION

Suppose that $f(z)$ is an entire function and that

$$M(r, f) = \sup_{|z|=r} |f(z)|$$

is the maximum modulus of $f(z)$. In this talk I should like to discuss how small $f(z)$ can become compared with $1/M(r)$ on a suitably large set E. Evidently $f(z) = 0$ at all the zeros of $f(z)$, so that we must not take E too small if we are to get a non-trivial result.

A classical problem concerns the minimum modulus

$$\mu(r, f) = \inf_{|z|=r} |f(z)| .$$

In our terminology this corresponds to a set E which meets every circle $|z| = r$. The quantity $\mu(r)$ was found to be important by Hadamard [2] in discussing the product representation for $f(z)$. We define the order λ or lower order μ of $f(z)$ by

$$\lambda = \overline{\lim_{r \to \infty}} \frac{\log \log M(r)}{\log r}, \quad \mu = \underline{\lim_{r \to \infty}} \frac{\log \log M(r)}{\log r} .$$

Let us suppose that $0 \leqslant \lambda < \infty$, let q be the integral part of λ and write

$$E(z, q) = (1-z) \exp \left\{ z + \frac{1}{2} z^2 + \ldots + \frac{1}{q} z^q \right\} .$$

If $f(z)$ has a zero of order p at the origin and other zeros z_ν each counted with correct multiplicity, we write

$$\Pi(z) = z^p \prod_{\nu=1}^{\infty} E\left(\frac{z}{z_\nu}, q\right) .$$

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[1]) Communicated to an International Symposium on Analysis, held in honour of Professor Albert Pfluger, ETH Zürich, 1978.

Then Hadamard showed that $\Pi(z)$ has order at most λ and

$$\mu(r, \Pi) > \exp(-r^{\lambda+\varepsilon})$$

for "most" values of r. Thus for such r

$$|F(z)| = \left|\frac{f(z)}{\Pi(z)}\right| < \exp(r^{\lambda+\varepsilon}), \quad |z| = r$$

and $F(z)$ is an entire function without zeros. Thus

$$F(z) = e^{P(z)}$$

where

$$Re\{P(z)\} < r^{\lambda+\varepsilon}, \quad |z| = r$$

and from this it is not difficult to show that $P(z)$ is a polynomial of degree at most λ. This yields the Hadamard product decomposition

(1) $$f(z) = e^{P(z)}\,\Pi(z).$$

The representation (1) is particularly useful when $\lambda < 1$, i.e. $q = 0$, when we obtain

(2) $$f(z) = az^p \prod_{\nu=1}^{\infty}\left(1 - \frac{z}{z_\nu}\right),$$

so that $f(z)$ has infinitely many zeros, unless f is a polynomial. It is also easy to deduce from (2) that the order λ depends only on the moduli of the zeros. Thus if we write $r_\nu = |z_\nu|$,

(3) $$F(z) = |a|\,z^p \prod_{\nu=1}^{\infty}\left(1 + \frac{z}{r_\nu}\right)$$

we deduce that

(4) $$|F(-r)| \leqslant \mu(r, f) \leqslant M(r, f) \leqslant F(r)$$

and

$$|F(r)F(-r)| \leqslant \mu(r, f)\,M(r, f).$$

These inequalities enable us to reduce the problem of the behaviour of functions of order less than one in most cases to that of the functions (3) which have all their zeros on the negative axis. Thus Valiron [9] and Wiman [10] proved the sharp result

(5) $$\mu(r) > M(r)^{\cos(\pi\lambda)-\varepsilon}$$

for a sequence of r tending to ∞. This had been conjectured by Littlewood [8] who proved the corresponding theorem with $\cos(2\pi\lambda)$ instead of $\cos(\pi\lambda)$. The result is valid for $0 \leqslant \lambda \leqslant 1$.

If $1 < \lambda < \infty$, Littlewood [8] also proved that there exists a positive constant $C(\lambda)$ such that

$$\mu(r) > M(r)^{-C(\lambda)-\varepsilon}$$

for a sequence of r tending to ∞. However the correct value of $C(\lambda)$ is unknown for $\lambda > 1$. It turns out that the formula (1) with exponential factors is much harder to work with than (2). Wiman [11] conjectured that $C(\lambda) = 1$ for $\lambda > 1$, a result which is true if $f(z)$ has no zeros. Later Beurling [1] proved a corresponding theorem for the case when $f(z)$ attains its minimum on a ray. Nevertheless Wiman's conjecture is false and the correct order of magnitude of Littlewood's constant $C(\lambda)$ is $\log \lambda$ as $\lambda \to \infty$. For infinite order the corresponding Theorem is [4].

$$\mu(r) > M(r)^{-A \log\log\log M(r)}, \tag{6}$$

where the best value of A lies between .09 and 11.03.

Since the theory of $\mu(r)$ is thus rather unsatisfactory for $\lambda > 1$ it is natural to consider other cases of E. Suppose first that E is a ray $\arg z = \theta$ and that $K > 1$. Then Beurling [1] showed that if

$$|f(re^{i\theta})| < M(r)^{-K}, \tag{7}$$

for $0 < r < R$, we have

$$|f(z)| < 1, \quad |z| = C_1(K)R,$$

where the constant $C_1(K)$ depends only on K. If R can be chosen arbitrarily large, we deduce at once that $f(z)$ is bounded on a sequence of large circles $|z| = C_1 R$, so that f is constant by Liouville's theorem. Thus for non-constant f (7) cannot be true for all r (or all large r) and a fixed θ.

2. The case when E is a curve

It is natural to consider the case when E is an unbounded connected set or equivalently a curve going to ∞ and this is the topic I mainly wish to discuss today. By a rather involved method I had shown [4] that in this case

$$|f(z)| > M(r)^{-A_0}, \tag{8}$$