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$> \sigma_1 - \varepsilon$ . If  $w$  is defined by  $w(z) = z + e^{i\theta}/z$  and  $f = w \circ h^{-1}$ , then  $f$  is univalent in  $A$  and

$$\|S_f\|_A = \|S_w - S_h\|_D \geq |S_w(0) - S_h(0)| = |6e^{i\theta} + S_h(0)|.$$

By choosing  $\varphi$  suitably we obtain  $\|S_f\|_A > 6 + \sigma_1 - \varepsilon$ .

## 5. SCHWARZIAN DERIVATIVE AND UNIVALENCE

5.1 *Constant  $\sigma_3$ .* Let  $A$  again be a simply connected domain with more than one boundary point. As a kind of opposite to the constant  $\sigma_2$  we define

$$\sigma_3 = \sup \{a \mid \|S_f\| \leq a \text{ implies } f \text{ univalent in } A\}.$$

Note that the number  $a = 0$  is always in the above set. In this definition, sup can be replaced by max, as can be shown by a standard normal family argument.

Nehari [12] proved that in a disc, the condition  $\|S_f\| \leq 2$  implies the univalence of  $f$ , and Hille [5] showed that the bound 2 is best possible. In other words,  $\sigma_3 = 2$  for a disc.

A closer study of  $\sigma_3$  leads to the universal Teichmüller space and reveals an intrinsic significance of quasiconformal mappings in the theory of univalent functions. The gist is the following result.

**THEOREM 5.1.** *The constant  $\sigma_3$  is positive if and only if  $A$  is bounded by a quasicircle.*

*Proof:* The sufficiency of the condition was established by Ahlfors [1] who actually proved more: If  $A$  is bounded by a  $K$ -quasicircle, there is an  $\varepsilon > 0$  depending only on  $K$ , such that whenever  $\|S_f\|_A < \varepsilon$ , then  $f$  is univalent and can be continued to a quasiconformal mapping of the plane. In the proof, the extension of the given meromorphic  $f$  is explicitly constructed by means of a continuously differentiable quasiconformal reflection  $\varphi$  in  $\partial A$  with bounded  $|d\varphi|/|dz|$  (cf. 3.3).

The necessity was proved by Gehring [2]. His proof was in two steps. It was first shown, by aid of an example, that if  $A$  is not  $b$ -locally connected for any  $b$ , then  $\sigma_3 = 0$ . After this, the desired conclusion was drawn from the result we stated above as Lemma 3.2.

5.2 *Universal Teichmüller space.* Henceforth, we assume that the domain  $A$  is bounded by a quasicircle. Let  $Q(A)$  be the Banach space

consisting of all holomorphic functions  $\varphi$  of  $A$  with finite norm. We introduce the subsets

$$U(A) = \{\varphi = S_f \mid f \text{ univalent in } A\},$$

$$T(A) = \{S_f \in U(A) \mid f \text{ can be extended to a quasiconformal mapping of the plane}\}.$$

Both sets are well defined. The set  $T(A)$  is called the *universal Teichmüller space* of  $A$ .

**THEOREM 5.2.** *The sets  $T(A)$  and  $U(A)$  are connected by the relation  $T(A) = \text{interior of } U(A)$ .*

*Proof:* We first show that  $T(A)$  is open. Choose  $S_f \in T(A)$ ,  $S_h \in Q(A)$ , and set  $g = h \circ f^{-1}$ . Then  $g$  is meromorphic in the domain  $f(A)$ . Since  $\partial A$  is a quasicircle,  $\partial f(A)$  is also a quasicircle. By the theorem of Ahlfors cited in the proof of Theorem 5.1, there is an  $\varepsilon > 0$  such that if

$$(5.1) \quad \|S_g\|_{f(A)} < \varepsilon,$$

then  $S_g \in T(f(A))$ . Now, choose  $h$  so that  $\|S_f - S_h\|_A < \varepsilon$ . Then (5.1) holds, and it follows that  $S_h = S_{g \circ f} \in T(A)$ .

After this it suffices to prove that  $\text{int } U(A) \subset T(A)$ . Choose  $S_f \in \text{int } U(A)$  and then an  $\varepsilon > 0$ , so that the ball  $B = \{\varphi \in Q(A) \mid \|\varphi - S_f\| < \varepsilon\}$  is contained in  $U(A)$ . Let  $g$  be an arbitrary meromorphic function in  $f(A)$  for which  $\|S_g\|_{f(A)} < \varepsilon$ . If  $h = g \circ f$ , then  $\|S_f - S_h\|_A = \|S_g\|_{f(A)} < \varepsilon$ . Thus  $S_h \in U(A)$ . But then also  $g = h \circ f^{-1}$  is univalent, and we have proved that  $\sigma_3$  is positive for the domain  $f(A)$ . By Theorem 5.1, the boundary  $\partial f(A)$  is a quasicircle. Hence, by the remark in 3.3,  $S_f \in T(A)$ .

**COROLLARY 5.1.** *If  $f$  is univalent in  $A$  and  $\|S_f\|_A < \sigma_3$ , then  $f$  can be extended to a quasiconformal mapping of the plane.*

*Proof:* This follows immediately from Theorem 5.2, in view of our previous remark that the closed ball  $\{\varphi \in Q(A) \mid \|\varphi\|_A \leq \sigma_3\}$  is contained in  $U(A)$ .

By this Corollary, we have for  $A$ ,

$$\sigma_3 = \sup \{a \mid \|S_f\|_A < a \text{ implies that } f \text{ is univalent and can be extended to a quasiconformal mapping of the plane}\}.$$

5.3 *New characterization for  $\sigma_3$ .* Theorem 5.2 was proved by Gehring [2] in the case where  $A$  is a half-plane. As is seen from the above proof, the generalization for an arbitrary  $A$  is immediate. In fact, the sets  $Q(A)$ ,  $U(A)$  and  $T(A)$  corresponding to different domains  $A$  are isomorphic:

LEMMA 5.1. *Let  $h$  be a conformal mapping of the upper half-plane  $H$  onto  $A$ . Then the mapping  $h^*: Q(A) \rightarrow Q(H)$ , defined by  $h^*(S_f) = S_{f \circ h}$ , is a bijective isometry. It maps  $U(A)$  and  $T(A)$  onto  $U(H)$  and  $T(H)$ , respectively.*

*Proof:* Clearly  $h^*$  is well defined and a bijection of  $Q(A)$ ,  $U(A)$  and  $T(A)$  onto  $Q(H)$ ,  $U(H)$  and  $T(H)$ , respectively. That  $h^*$  is an isometry follows from formula (4.3).

The function  $h^*$  maps the origin of  $Q(A)$  onto the point  $S_h \in T(H)$ , which has the distance  $\sigma_1$  from the origin of  $Q(H)$ . If  $B = \{\varphi \in Q(A) \mid \|\varphi\|_A \leq \sigma_3\}$ , then

$$h^*(B) = \{\psi \in Q(H) \mid \|\psi - S_h\| \leq \sigma_3\}.$$

From this and the definition of  $\sigma_3$  we infer that  $\sigma_3$  is equal to the distance from the point  $S_h$  to the boundary of  $U(H)$ . The following characterization seems to be more useful:

LEMMA 5.2. *The constant  $\sigma_3$  of  $A$  is equal to the distance of the point  $S_h$  to the boundary of  $T(H)$ .*

*Proof:* Let  $d$  denote the distance function in  $Q$ . Since  $T(H) \subset U(H)$  we conclude from what we just said above that  $\sigma_3 \geq d(\{S_h\}, U(H) - T(H))$ . On the other hand, it follows from Theorem 5.2 that  $\text{int } B \subset T(A)$  and hence  $\text{int } h^*(B) \subset T(H)$ . Therefore,  $\sigma_3 \leq d(\{S_h\}, U(H) - T(H))$ .

A standard normal family argument shows that  $U(A)$  is a closed subset of  $Q(A)$ . Therefore, the closure of  $T(A)$  is contained in  $U(A)$ . Gehring [3] showed recently that this inclusion is proper, thus disproving a famous conjecture of Bers.

However, it is true that on every sphere  $\|\varphi\| = r$  of  $Q(H)$ ,  $2 \leq r \leq 6$ , there are points of  $U(H) - T(H)$  which belong to the closure of  $T(H)$  ([9]).

5.4 *Estimates for  $\sigma_3$ .* Lemma 5.2 can be used to deriving estimates for  $\sigma_3$  in terms of  $\sigma_1$  ([9]). Suppose first that  $0 \leq \sigma_1 < 2$ . Then  $S_h$  lies in the ball  $\{\varphi \in Q(H) \mid \|\varphi\| < 2\}$  which is a subset of  $T(H)$ . Since  $\|S_h\| = \sigma_1$ ,

we conclude that  $d(\{S_h\}, U(H) - T(H)) \geq 2 - \sigma_1$ . Consequently, by Lemma 5.2,

$$(5.2) \quad \sigma_3 \geq 2 - \sigma_1.$$

In order to prove that this inequality is sharp, we consider the point  $S_w$ , where  $w$  is the restriction to  $H$  of a branch of the logarithm. Since the boundary of  $w(H)$  is not a quasicircle,  $S_w \in U(H) - T(H)$ . From  $S_w(z) = z^{-2}/2$  it follows that  $\|S_w\|_H = 2$ . Let  $h$  be determined by the condition  $S_h = r S_w$ ,  $0 < r < 1$ , and set  $A = h(H)$ . From  $\|S_h\|_H < 2$  it follows that  $S_h \in T(H)$ , and so  $\partial A$  is a quasicircle. Now

$$\sigma_3 = d(\{S_h\}, U(H) - T(H)) = \|S_w - S_h\| = 2(1-r) = 2 - \sigma_1,$$

showing that (5.2) is sharp.

Suppose that  $2 \leq \sigma_1 < 6$ . We then conclude from the remark at the end of 5.3 that, even though  $\sigma_3 > 0$  for each  $A$ , we have  $\inf \sigma_3 = 0$  for every  $\sigma_1$ .

Similarly, Lemma 5.2 can be used to deriving the upper estimate

$$\sigma_3 \leq \min(2, 6 - \sigma_1).$$

(For the details we refer to [9].)

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