Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 24 (1978)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: UNIVALENT FUNCTIONS, SCHWARZIAN DERIVATIVES AND QUASICONFORMAL MAPPINGS

Autor: Lehto, Olli

Kapitel: 5. SCHWARZIAN DERIVATIVE AND UNIVALENCE

DOI: https://doi.org/10.5169/seals-49701

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 16.10.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

$> \sigma_1 - \varepsilon$. If w is defined by $w(z) = z + e^{i\theta} / z$ and $f = w \circ h^{-1}$, then f is univalent in A and

$$\| S_f \|_A = \| S_w - S_h \|_D \geqslant | S_w(0) - S_h(0) | = | 6e^{i\theta} + S_h(0) | .$$

By choosing φ suitably we obtain $\| S_f \|_A > 6 + \sigma_1 - \varepsilon$.

5. Schwarzian derivative and univalence

5.1 *Constant* σ_3. Let A again be a simply connected domain with more than one boundary point. As a kind of opposite to the constant σ_2 we define

$$\sigma_3 = \sup \{a \mid \| S_f \| \leqslant a \text{ implies } f \text{ univalent in } A\}.$$

Note that the number $a = 0$ is always in the above set. In this definition, sup can be replaced by max, as can be shown by a standard normal family argument.

Nehari [12] proved that in a disc, the condition $\| S_f \| \leqslant 2$ implies the univalence of f, and Hille [5] showed that the bound 2 is best possible. In other words, $\sigma_3 = 2$ for a disc.

A closer study of σ_3 leads to the universal Teichmüller space and reveals an intrinsic significance of quasiconformal mappings in the theory of univalent functions. The gist is the following result.

Theorem 5.1. *The constant* σ_3 *is positive if and only if* A *is bounded by a quasicircle.*

Proof: The sufficiency of the condition was established by Ahlfors [1] who actually proved more: If A is bounded by a K-quasicircle, there is an $\varepsilon > 0$ depending only on K, such that whenever $\| S_f \|_A < \varepsilon$, then f is univalent and can be continued to a quasiconformal mapping of the plane. In the proof, the extension of the given meromorphic f is explicitly constructed by means of a continuously differentiable quasiconformal reflection φ in ∂A with bounded $| d\varphi | / | dz |$ (cf. 3.3).

The necessity was proved by Gehring [2]. His proof was in two steps. It was first shown, by aid of an example, that if A is not b-locally connected for any b, then $\sigma_3 = 0$. After this, the desired conclusion was drawn from the result we stated above as Lemma 3.2.

5.2 *Universal Teichmüller space.* Henceforth, we assume that the domain A is bounded by a quasicircle. Let $Q(A)$ be the Banach space

consisting of all holomorphic functions φ of A with finite norm. We introduce the subsets

$$U(A) = \{\varphi = S_f \mid f \text{ univalent in } A\},$$
$$T(A) = \{S_f \in U(A) \mid f \text{ can be extended to a quasiconformal mapping of the plane}\}.$$

Both sets are well defined. The set $T(A)$ is called the *universal Teichmüller space* of A.

THEOREM 5.2. *The sets* $T(A)$ *and* $U(A)$ *are connected by the relation* $T(A) =$ interior of $U(A)$.

Proof: We first show that $T(A)$ is open. Choose $S_f \in T(A)$, $S_h \in Q(A)$, and set $g = h \circ f^{-1}$. Then g is meromorphic in the domain $f(A)$. Since ∂A is a quasicircle, $\partial f(A)$ is also a quasicircle. By the theorem of Ahlfors cited in the proof of Theorem 5.1, there is an $\varepsilon > 0$ such that if

$$\| S_g \|_{f(A)} < \varepsilon, \tag{5.1}$$

then $S_g \in T(f(A))$. Now, choose h so that $\| S_f - S_h \|_A < \varepsilon$. Then (5.1) holds, and it follows that $S_h = S_{g \circ f} \in T(A)$.

After this it suffices to prove that int $U(A) \subset T(A)$. Choose $S_f \in$ int $U(A)$ and then an $\varepsilon > 0$, so that the ball $B = \{\varphi \in Q(A) \mid \| \varphi - S_f \| < \varepsilon\}$ is contained in $U(A)$. Let g be an arbitrary meromorphic function in $f(A)$ for which $\| S_g \|_{f(A)} < \varepsilon$. If $h = g \circ f$, then $\| S_f - S_h \|_A = \| S_g \|_{f(A)} < \varepsilon$. Thus $S_h \in U(A)$. But then also $g = h \circ f^{-1}$ is univalent, and we have proved that σ_3 is positive for the domain $f(A)$. By Theorem 5.1, the boundary $\partial f(A)$ is a quasicircle. Hence, by the remark in 3.3, $S_f \in T(A)$.

COROLLARY 5.1. *If* f *is univalent in* A *and* $\| S_f \|_A < \sigma_3$, *then* f *can be extended to a quasiconformal mapping of the plane.*

Proof: This follows immediately from Theorem 5.2, in view of our previous remark that the closed ball $\{\varphi \in Q(A) \mid \| \varphi \|_A \leqslant \sigma_3\}$ is contained in $U(A)$.

By this Corollary, we have for A,

$$\sigma_3 = \sup \{a \mid \| S_f \|_A < a \text{ implies that } f \text{ is univalent and can be extended to a quasiconformal mapping of the plane}\}.$$

5.3 *New characterization for* σ_3. Theorem 5.2 was proved by Gehring [2] in the case where A is a half-plane. As is seen from the above proof, the generalization for an arbitrary A is immediate. In fact, the sets $Q(A)$, $U(A)$ and $T(A)$ corresponding to different domains A are isomorphic:

LEMMA 5.1. *Let h be a conformal mapping of the upper half-plane H onto A. Then the mapping $h^*: Q(A) \to Q(H)$, defined by $h^*(S_f) = S_{f \circ h}$, is a bijective isometry. It maps $U(A)$ and $T(A)$ onto $U(H)$ and $T(H)$, respectively.*

Proof: Clearly h^* is well defined and a bijection of $Q(A)$, $U(A)$ and $T(A)$ onto $Q(H)$, $U(H)$ and $T(H)$, respectively. That h^* is an isometry follows from formula (4.3).

The function h^* maps the origin of $Q(A)$ onto the point $S_h \in T(H)$, which has the distance σ_1 from the origin of $Q(H)$. If $B = \{\varphi \in Q(A) \mid \| \varphi \|_A \leqslant \sigma_3\}$, then

$$h^*(B) = \{\psi \in Q(H) \mid \| \psi - S_h \| \leqslant \sigma_3\}.$$

From this and the definition of σ_3 we infer that σ_3 is equal to the distance from the point S_h to the boundary of $U(H)$. The following characterization seems to be more useful:

LEMMA 5.2. *The constant σ_3 of A is equal to the distance of the point S_h to the boundary of $T(H)$.*

Proof: Let d denote the distance function in Q. Since $T(H) \subset U(H)$ we conclude from what we just said above that $\sigma_3 \geqslant d(\{S_h\}, U(H) - T(H))$. On the other hand, it follows from Theorem 5.2 that int $B \subset T(A)$ and hence int $h^*(B) \subset T(H)$. Therefore, $\sigma_3 \leqslant d(\{S_h\}, U(H) - T(H))$.

A standard normal family argument shows that $U(A)$ is a closed subset of $Q(A)$. Therefore, the closure of $T(A)$ is contained in $U(A)$. Gehring [3] showed recently that this inclusion is proper, thus disproving a famous conjecture of Bers.

However, it is true that on every sphere $\| \varphi \| = r$ of $Q(H)$, $2 \leqslant r \leqslant 6$, there are points of $U(H) - T(H)$ which belong to the closure of $T(H)$ ([9]).

5.4 *Estimates for* σ_3. Lemma 5.2 can be used to deriving estimates for σ_3 in terms of σ_1 ([9]). Suppose first that $0 \leqslant \sigma_1 < 2$. Then S_h lies in the ball $\{\varphi \in Q(H) \mid \| \varphi \| < 2\}$ which is a subset of $T(H)$. Since $\| S_h \| = \sigma_1$,

we conclude that $d(\{S_h\}, U(H)-T(H)) \geqslant 2-\sigma_1$. Consequently, by Lemma 5.2,

$$\sigma_3 \geqslant 2-\sigma_1 . \tag{5.2}$$

In order to prove that this inequality is sharp, we consider the point S_w, where w is the restriction to H of a branch of the logarithm. Since the boundary of $w(H)$ is not a quasicircle, $S_w \in U(H) - T(H)$. From $S_w(z) = z^{-2}/2$ it follows that $\| S_w \|_H = 2$. Let h be determined by the condition $S_h = r S_w$, $0 < r < 1$, and set $A = h(H)$. From $\| S_h \|_H < 2$ it follows that $S_h \in T(H)$, and so ∂A is a quasicircle. Now

$$\sigma_3 = d(\{S_h\}, U(H)-T(H)) = \| S_w - S_h \| = 2(1-r) = 2-\sigma_1 ,$$

showing that (5.2) is sharp.

Suppose that $2 \leqslant \sigma_1 < 6$. We then conclude from the remark at the end of 5.3 that, even though $\sigma_3 > 0$ for each A, we have $\inf \sigma_3 = 0$ for every σ_1.

Similarly, Lemma 5.2 can be used to deriving the upper estimate

$$\sigma_3 \leqslant \min (2, 6-\sigma_1) .$$

(For the details we refer to [9].)

References

[1] Ahlfors, L. V. Quasiconformal reflections. *Acta Math. 109* (1963), pp. 291-301.

[2] Gehring, F. W. Univalent functions and the Schwarzian derivative. *Comment. Math. Helv. 52* (1977), pp. 561-572.

[3] —— Spirals and the universal Teichmüller space. *To appear in Acta Math.*

[4] Gehring, F. W. and J. Väisälä. Hausdorff dimension and quasiconformal mappings. *J. London Math. Soc.* (1973).

[5] Hille, E. Remarks on a paper by Zeev Nehari. *Bull. Amer. Math. Soc. 55* (1949), pp. 552-553.

[6] Kraus, W. Über den Zusammenhang einiger Charakteristiken eines einfach zusammenhängenden Bereichs mit der Kreisabbildung. *Mitt. math. Semin. Giessen 21* (1932).

[7] Kühnau, R. Wertannahmeprobleme bei quasikonformen Abbildungen mit ortsabhängiger Dilatationsbeschränkung. *Math. Nachr. 40* (1969).

[8] Lehto, O. Schlicht functions with a quasiconformal extension. *Ann. Acad. Sci. Fenn. A I 500* (1971).

[9] —— Domain constants associated with Schwarzian derivative. *Comment. Math. Helv. 52* (1977), pp. 603-610.

[10] —— Univalent functions and Teichmüller theory. *Proc. of the First Finnish-Polish Summer School in Complex Analysis at Podlesice, University of Lodz* (1977), pp. 11-33.