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- 1° Any two points of the set  $E \cap \text{cl}U(z, r)$  can be joined by an arc lying in  $E \cap \text{cl}U(z, br)$ .
- 2° Any two points of the set  $E - U(z, r)$  can be joined by an arc lying in  $E - U(z, r/b)$ .

The following result has recently been proved by Gehring [2]:

**LEMMA 3.2.** *Let the set  $C$  contain at least two points and bound a simply connected domain  $A$ . If  $A$  is  $b$ -locally connected, then  $C$  is a  $c(b)$ -quasicircle, where  $c(b)$  depends only on  $b$ .*

**3.3 Quasiconformal reflection.** Let  $C$  be a Jordan curve bounding the domains  $A$  and  $B$ . A sense-reversing  $K$ -quasiconformal mapping  $\varphi: A \rightarrow B$  is a  $K$ -quasiconformal reflection in  $C$  if  $\varphi$  leaves every point of  $C$  invariant.

It is not difficult to prove that  $C$  admits a quasiconformal reflection if and only if  $C$  is a quasicircle. It follows that a quasiconformal mapping  $f: A \rightarrow B$  between domains  $A$  and  $B$  bounded by quasicircles can be extended to a quasiconformal mapping of the plane. In fact, if  $\varphi$  and  $\psi$  are quasiconformal reflections in the boundaries  $\partial A$  and  $\partial B$ , such that  $\varphi$  is defined outside  $A$  and  $\psi$  in  $B$ , then  $\psi \circ f \circ \varphi$  extends  $f$  quasiconformally.

A quasicircle always admits quasiconformal reflections which are continuously differentiable or even real-analytic. For a  $K$ -quasicircle passing through  $\infty$ , a reflection  $\varphi$  exists such that  $|d\varphi(z)| / |dz|$  is bounded by a constant depending only on  $K$ .

For more details of the properties of quasicircles we refer to [10].

#### 4. DEVIATION OF A DOMAIN FROM A DISC

**4.1 Schwarzian derivative.** Let  $f$  be a locally injective meromorphic function in a simply connected domain  $A$ . At finite points of  $A$  which are not poles of  $f$ , the *Schwarzian derivative*  $S_f$  of  $f$  is defined by

$$S_f = (f''/f')' - \frac{1}{2} (f''/f')^2,$$

and the definition is extended to  $\infty$  and to the poles of  $f$  by means of inversion.

The Schwarzian derivative is holomorphic in  $A$ . Conversely, every function which is holomorphic in  $A$  is the Schwarzian of some  $f$ . The Schwarzian vanishes identically if and only if  $f$  is a Möbius transformation.

More generally, the Schwarzian determines a function up to a Möbius transformation.

Suppose the boundary of  $A$  consists of more than one point; then a conformal mapping  $h$  of  $A$  onto the unit disc exists. Through  $h$  a conformally invariant metric  $\rho(z)|dz|$  is defined in  $A$ , by the rule  $\rho(z)|dz| = (1-|w|^2)^{-1}|dw|$ ,  $w = h(z)$ . For functions  $\varphi$  holomorphic in  $A$  we introduce the norm

$$\|\varphi\|_A = \sup_{z \in A} |\varphi(z)| \rho(z)^{-2}.$$

The Schwarzian obeys the composition rule  $S_{f \circ g} = (S_f \circ g)f'^2 + S_g$ . We note certain of its immediate consequences. First, let  $f$  be meromorphic in  $A$  and  $h: A \rightarrow B$  a conformal mapping. Then

$$(4.1) \quad |S_f(z) - S_h(z)| \rho_A(z)^{-2} = |S_{f \circ h^{-1}}(\zeta)| \rho_B(\zeta)^{-2}, \quad \zeta = h(z).$$

It follows that  $\|S_f - S_h\|_A = \|S_{f \circ h^{-1}}\|_B$ . In particular,

$$(4.2) \quad \|S_h\|_A = \|S_{h^{-1}}\|_B.$$

Secondly, let  $f$  and  $g$  be meromorphic in  $A$  and  $h: G \rightarrow A$  a conformal mapping. Then

$$(4.3) \quad \|S_{f \circ h} - S_{g \circ h}\|_G = \|S_f - S_g\|_A.$$

Finally, we remark that the norm of the Schwarzian is completely invariant under Möbius transformations: If  $f$  is meromorphic in  $A$  and  $g$  and  $h$  are Möbius transformations, then  $\|S_{h \circ f \circ g}\|_{g^{-1}(A)} = \|S_f\|_A$ .

**4.2 Constant  $\sigma_1$ .** We associate with the domain  $A$  the constant  $\sigma_1 = \|S_f\|_A$ , where  $f$  is a conformal map of  $A$  onto a disc. Here a disc means an ordinary disc or a half-plane. The number  $\sigma_1$  is well defined, and equal to 0 if and only if  $A$  itself is a disc. It can be regarded as a measure of how much the domain  $A$  differs from a disc.

It is well known that  $\sigma_1 \leq 6$  (Theorem of Kraus [6]). For the domain  $A = \{z \mid 0 < \arg z < k\pi\}$ ,  $1 \leq k \leq 2$ , we have  $\sigma_1 = 2(k^2 - 1)$ . It follows that  $\sigma_1$  can take any value in the closed interval  $[0, 6]$ .

**4.3 Domains bounded by a quasicircle.** In some cases, information about the boundary of  $A$  makes it possible to improve the estimate  $\sigma_1 \leq 6$ .

**THEOREM 4.1.** *For a domain  $A$  bounded by a  $K$ -quasicircle,*

$$(4.4) \quad \sigma_1 \leq 6 \frac{K^2 - 1}{K^2 + 1}.$$

*Proof:* By Lemma 3.1, there exists a  $K^2$ -quasiconformal mapping  $w$  of the plane whose restriction to the upper half-plane  $H$  maps  $H$  conformally onto  $A$ . For the function  $w|H$  the Krauss estimate can be improved:

$$\|S_{w|H}\|_H \leq 6 \frac{K^2 - 1}{K^2 + 1};$$

for the proof we refer to Kühnau [7], or to [8]. Hence (4.4) follows from (4.2).

**4.4 Domains with bounded boundary rotation.** Let  $A$  be a domain bounded by a continuously differentiable Jordan curve. The total variation of the direction angle of the boundary tangent under a complete circuit is called the boundary rotation of  $A$ . If the boundary is not so regular, boundary rotation is defined by means of approximations from inside.

Let  $f$  be a conformal mapping of the unit disc  $D$  onto a domain  $A$  with boundary rotation  $k\pi$ ,  $2 \leq k < \infty$ . A real-valued function  $\psi$  with the properties

$$\int_0^{2\pi} d\psi(\theta) = 2, \quad \int_0^{2\pi} |d\psi(\theta)| = k,$$

can be associated with  $f$ , such that

$$(4.5) \quad f'(z) = f'(0) \exp \left( - \int_0^{2\pi} \log (1 - ze^{-i\theta}) d\psi(\theta) \right).$$

The domain  $A$  is convex if and only if  $k = 2$ . This is equivalent to  $\psi$  being an increasing function. A function  $f$  whose derivative admits the representation (4.5) is always univalent if the total variation of  $\psi$  is  $\leq 4$ .

Domains with bounded boundary rotation were introduced by Löwner and their basic properties established by Paatero [14].

**THEOREM 4.2.** *For a domain  $A$  with boundary rotation  $\leq k\pi$ ,  $2 \leq k \leq 4$ ,*

$$(4.6) \quad \sigma_1 \leq \frac{2k + 4}{6 - k}.$$

*The bound is sharp.*

*Proof:* Let  $f: D \rightarrow A$  be a conformal mapping,  $z_0$  an arbitrary point of  $D$ , and  $h$  a conformal self-mapping of  $D$ , such that  $h(0) = z_0$ . Since  $\rho_D(0) = 1$ , it follows from (4.1) that

(4.7)

$$|S_f(z_0)|\rho_D(z_0)^{-2} = |S_{f \circ h}(0)|.$$

Hence, (4.6) follows if we prove that  $|S_f(0)| \leq (2k+4)/(6-k)$ . Since we may replace  $f$  by the function  $z \rightarrow cf(ze^{i\varphi})$ ,  $c$  complex,  $\varphi$  real, there is no loss of generality in assuming that  $S_f(0) \geq 0$  and that  $f'(0) = 1$ . From the representation formula (4.5) we then deduce that

$$(4.8) \quad S_f(0) = \int_0^{2\pi} \cos 2\theta d\psi(\theta) - \frac{1}{2} \left( \int_0^{2\pi} \cos \theta d\psi(\theta) \right)^2 + \frac{1}{2} \left( \int_0^{2\pi} \sin \theta d\psi(\theta) \right)^2.$$

If  $k = 2$ , we have  $d\psi(\theta) \geq 0$ . In this case we get the inequality  $\sigma_1 \leq 2$  for convex domains from (4.8) quite easily, just by use of Schwarz's inequality. Extremal functions can also be determined. These computations have been carried out in [9]. Nehari [13] proved the result  $\sigma_1 \leq 2$  by means of variational methods.

If  $2 < k \leq 4$ , establishing (4.6) requires a more careful handling of formula (4.8). These computations will be published in a joint paper with O. Tammi.

Matti Lehtinen has let me know that for functions  $f$  whose derivative satisfies (4.5) with a  $\psi$  whose total variation is  $\leq k$ ,  $k \geq 4$ , the sharp upper bound for  $\|S_f\|$  is equal to  $(k^2 - 4)/2$ . The extremal functions are not univalent.

#### 4.5 Constant $\sigma_2$ . The domain constant

$$\sigma_2 = \sup \{ \|S_f\|_A \mid f \text{ univalent in } A \}$$

is in simple relation with  $\sigma_1$  ([9]):

**THEOREM 4.3.** *In every domain  $A$ ,  $\sigma_2 = \sigma_1 + 6$ .*

*Proof:* Let  $f$  be univalent in  $A$  and  $h: D \rightarrow A$  conformal. By (4.3),

$$\|S_f\|_A = \|S_{f \circ h} - S_f\|_D \leq 6 + \|S_h\|_D = 6 + \sigma_1.$$

In order to derive an estimate in the opposite direction, let an  $\varepsilon > 0$  be given. In view of formula (4.7), we can choose  $h: D \rightarrow A$  so that  $|S_h(0)|$

$> \sigma_1 - \varepsilon$ . If  $w$  is defined by  $w(z) = z + e^{i\theta}/z$  and  $f = w \circ h^{-1}$ , then  $f$  is univalent in  $A$  and

$$\|S_f\|_A = \|S_w - S_h\|_D \geq |S_w(0) - S_h(0)| = |6e^{i\theta} + S_h(0)|.$$

By choosing  $\varphi$  suitably we obtain  $\|S_f\|_A > 6 + \sigma_1 - \varepsilon$ .

## 5. SCHWARZIAN DERIVATIVE AND UNIVALENCE

5.1 *Constant  $\sigma_3$ .* Let  $A$  again be a simply connected domain with more than one boundary point. As a kind of opposite to the constant  $\sigma_2$  we define

$$\sigma_3 = \sup \{a \mid \|S_f\| \leq a \text{ implies } f \text{ univalent in } A\}.$$

Note that the number  $a = 0$  is always in the above set. In this definition,  $\sup$  can be replaced by  $\max$ , as can be shown by a standard normal family argument.

Nehari [12] proved that in a disc, the condition  $\|S_f\| \leq 2$  implies the univalence of  $f$ , and Hille [5] showed that the bound 2 is best possible. In other words,  $\sigma_3 = 2$  for a disc.

A closer study of  $\sigma_3$  leads to the universal Teichmüller space and reveals an intrinsic significance of quasiconformal mappings in the theory of univalent functions. The gist is the following result.

**THEOREM 5.1.** *The constant  $\sigma_3$  is positive if and only if  $A$  is bounded by a quasicircle.*

*Proof:* The sufficiency of the condition was established by Ahlfors [1] who actually proved more: If  $A$  is bounded by a  $K$ -quasicircle, there is an  $\varepsilon > 0$  depending only on  $K$ , such that whenever  $\|S_f\|_A < \varepsilon$ , then  $f$  is univalent and can be continued to a quasiconformal mapping of the plane. In the proof, the extension of the given meromorphic  $f$  is explicitly constructed by means of a continuously differentiable quasiconformal reflection  $\varphi$  in  $\partial A$  with bounded  $|d\varphi|/|dz|$  (cf. 3.3).

The necessity was proved by Gehring [2]. His proof was in two steps. It was first shown, by aid of an example, that if  $A$  is not  $b$ -locally connected for any  $b$ , then  $\sigma_3 = 0$ . After this, the desired conclusion was drawn from the result we stated above as Lemma 3.2.

5.2 *Universal Teichmüller space.* Henceforth, we assume that the domain  $A$  is bounded by a quasicircle. Let  $\mathcal{Q}(A)$  be the Banach space