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- 1° Any two points of the set  $E \cap clU(z, r)$  can be joined by an arc lying in  $E \cap clU(z, br)$ .
- 2° Any two points of the set E U(z, r) can be joined by an arc lying in E U(z, r/b).

The following result has recently been proved by Gehring [2]:

LEMMA 3.2. Let the set C contain at least two points and bound a simply connected domain A. If A is b-locally connected, then C is a c(b)-quasicircle, where c(b) depends only on b.

3.3 Quasiconformal reflection. Let C be a Jordan curve bounding the domains A and B. A sense-reversing K-quasiconformal mapping  $\varphi: A \to B$  is a K-quasiconformal reflection in C if  $\varphi$  leaves every point of C invariant.

It is not difficult to prove that C admits a quasiconformal reflection if and only if C is a quasicircle. It follows that a quasiconformal mapping  $f: A \to B$  between domains A and B bounded by quasicircles can be extended to a quasiconformal mapping of the plane. In fact, if  $\varphi$  and  $\psi$  are quasiconformal reflections in the boundaries  $\partial A$  and  $\partial B$ , such that  $\varphi$  is defined outside A and  $\psi$  in B, then  $\psi \circ f \circ \varphi$  extends f quasiconformally.

A quasicircle always admits quasiconformal reflections which are continuously differentiable or even real-analytic. For a K-quasicircle passing through  $\infty$ , a reflection  $\varphi$  exists such that  $|d\varphi(z)|/|dz|$  is bounded by a constant depending only on K.

For more details of the properties of quasicircles we refer to [10].

## 4. DEVIATION OF A DOMAIN FROM A DISC

4.1 Schwarzian derivative. Let f be a locally injective meromorphic function in a simply connected domain A. At finite points of A which are not poles of f, the Schwarzian derivative  $S_f$  of f is defined by

$$S_f = (f''/f')' - \frac{1}{2} (f''/f')^2,$$

and the definition is extended to  $\infty$  and to the poles of f by means of inversion.

The Schwarzian derivative is holomorphic in A. Conversely, every function which is holomorphic in A is the Schwarzian of some f. The Schwarzian vanishes identically if and only if f is a Möbius transformation.

More generally, the Schwarzian determines a function up to a Möbius transformation.

Suppose the boundary of A consists of more than one point; then a conformal mapping h of A onto the unit disc exists. Through h a conformally invariant metric  $\rho(z) | dz |$  is defined in A, by the rule  $\rho(z) | dz | = (1 - |w|^2)^{-1} |dw|, w = h(z)$ . For functions  $\varphi$  holomorphic in A we introduce the norm

$$\|\varphi\|_{A} = \sup_{z \in A} |\varphi(z)| \rho(z)^{-2}.$$

The Schwarzian obeys the composition rule  $S_{f \circ g} = (S_f \circ g) f'^2 + S_g$ . We note certain of its immediate consequences. First, let f be meromorphic in A and  $h: A \to B$  a conformal mapping. Then

(4.1) 
$$|S_f(z) - S_h(z)| \rho_A(z)^{-2} = |S_{f^{\circ h^{-1}}}(\zeta)| \rho_B(\zeta)^{-2}, \quad \zeta = h(z).$$

It follows that  $|| S_f - S_h ||_A = || S_{f^{\circ h^{-1}}} ||_B$ . In particular,

(4.2) 
$$|| S_h ||_A = || S_{h^{-1}} ||_B.$$

Secondly, let f and g be meromorphic in A and  $h: G \rightarrow A$  a conformal mapping. Then

(4.3) 
$$|| S_{f \circ h} - S_{g \circ h} ||_G = || S_f - S_g ||_A.$$

Finally, we remark that the norm of the Schwarzian is completely invariant under Möbius transformations: If f is meromorphic in A and g and h are Möbius transformations, then  $|| S_{h \circ f \circ g} ||_{g^{-1}(A)} = || S_f ||_A$ .

4.2 Constant  $\sigma_1$ . We associate with the domain A the constant  $\sigma_1 = \|S_f\|_A$ , where f is a conformal map of A onto a disc. Here a disc means an ordinary disc or a half-plane. The number  $\sigma_1$  is well defined, and equal to 0 if and only if A itself is a disc. It can be regarded as a measure of how much the domain A differs from a disc.

It is well known that  $\sigma_1 \leq 6$  (Theorem of Kraus [6]). For the domain  $A = \{z \mid 0 < \arg z < k\pi\}, 1 \leq k \leq 2$ , we have  $\sigma_1 = 2(k^2-1)$ . If follows that  $\sigma_1$  can take any value in the closed interval [0, 6].

4.3 Domains bounded by a quasicircle. In some cases, information about the boundary of A makes it possible to improve the estimate  $\sigma_1 \leq 6$ .

THEOREM 4.1. For a domain A bounded by a K-quasicircle,

(4.4) 
$$\sigma_1 \leqslant 6 \frac{K^2 - 1}{K^2 + 1}.$$

*Proof*: By Lemma 3.1, there exists a  $K^2$ -quasiconformal mapping w of the plane whose restriction to the upper half-plane H maps H conformally onto A. For the function w | H the Krauss estimate can be improved:

$$\|S_{w|H}\|_{H} \leqslant 6 \frac{K^{2}-1}{K^{2}+1};$$

for the proof we refer to Kühnau [7], or to [8]. Hence (4.4) follows from (4.2).

4.4 Domains with bounded boundary rotation. Let A be a domain bounded by a continuously differentiable Jordan curve. The total variation of the direction angle of the boundary tangent under a complete circuit is called the boundary rotation of A. If the boundary is not so regular, boundary rotation is defined by means of approximations from inside.

Let f be a conformal mapping of the unit disc D onto a domain A with boundary rotation  $k\pi$ ,  $2 \le k < \infty$ . A real-valued function  $\psi$  with the properties

$$\int_{0}^{2\pi} d\psi(\theta) = 2, \int_{0}^{2\pi} |d\psi(\theta)| = k,$$

can be associated with f, such that

(4.5) 
$$f'(z) = f'(0) \exp\left(-\int_{0}^{2\pi} \log(1-ze^{-i\theta}) d\psi(\theta)\right).$$

The domain A is convex if and only if k = 2. This is equivalent to  $\psi$  being an increasing function. A function f whose derivative admits the representation (4.5) is always univalent if the total variation of  $\psi$  is  $\leq 4$ .

Domains with bounded boundary rotation were introduced by Löwner and their basic properties established by Paatero [14].

THEOREM 4.2. For a domain A with boundary rotation  $\leq k\pi$ ,  $2 \leq k \leq 4$ ,

(4.6) 
$$\sigma_1 \leqslant \frac{2k+4}{6-k}.$$

The bound is sharp.

*Proof*: Let  $f: D \to A$  be a conformal mapping,  $z_0$  an arbitrary point of D, and h a conformal self-mapping of D, such that  $h(0) = z_0$ . Since  $\rho_D(0) = 1$ , it follows from (4.1) that - 209 —

(4.7) 
$$|S_f(z_0)| \rho_D(z_0)^{-2} = |S_{foh}(0)|.$$

Hence, (4.6) follows if we prove that  $|S_f(0)| \leq (2k+4)/(6-k)$ . Since we may replace f by the function  $z \to cf(ze^{i\varphi})$ , c complex,  $\varphi$  real, there is no loss of generality in assuming that  $S_f(0) \geq 0$  and that f'(0) = 1. From the representation formula (4.5) we then deduce that

(4.8) 
$$S_{f}(0) = \int_{0}^{2\pi} \cos 2\theta \, d\psi(\theta) - \frac{1}{2} \left( \int_{0}^{2\pi} \cos\theta d\psi(\theta) \right)^{2} + \frac{1}{2} \left( \int_{0}^{2\pi} \sin\theta d\psi(\theta) \right)^{2}.$$

If k = 2, we have  $d\psi(\theta) \ge 0$ . In this case we get the inequality  $\sigma_1 \le 2$  for convex domains from (4.8) quite easily, just by use of Schwarz's inequality. Extremal functions can also be determined. These computations have been carried out in [9]. Nehari [13] proved the result  $\sigma_1 \le 2$  by means of variational methods.

If  $2 < k \le 4$ , establishing (4.6) requires a more careful handling of formula (4.8). These computations will be published in a joint paper with O. Tammi.

Matti Lehtinen has let me know that for functions f whose derivative satisfies (4.5) with a  $\psi$  whose total variation is  $\leq k, k \geq 4$ , the sharp upper bound for  $||S_f||$  is equal to  $(k^2 - 4)/2$ . The extremal functions are not univalent.

4.5 Constant  $\sigma_2$ . The domain constant

 $\sigma_2 = \sup \{ \| S_f \|_A | f \text{ univalent in } A \}$ 

is in simple relation with  $\sigma_1$  ([9]):

THEOREM 4.3. In every domain A,  $\sigma_2 = \sigma_1 + 6$ .

*Proof*: Let f be univalent in A and  $h: D \to A$  conformal. By (4.3),

$$\|S_f\|_A = \|S_{f \circ h} - S_f\|_D \leqslant 6 + \|S_h\|_D = 6 + \sigma_1.$$

In order to derive an estimate in the opposite direction, let an  $\varepsilon > 0$  be given. In view of formula (4.7), we can choose  $h: D \to A$  so that  $|S_h(0)|$ 

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 $> \sigma_1 - \varepsilon$ . If w is defined by  $w(z) = z + e^{i\theta} / z$  and  $f = w \circ h^{-1}$ , then f is univalent in A and

 $\|S_f\|_A = \|S_w - S_h\|_D \ge |S_w(0) - S_h(0)| = |6e^{i\theta} + S_h(0)|.$ 

By choosing  $\varphi$  suitably we obtain  $\|S_f\|_A > 6 + \sigma_1 - \varepsilon$ .

# 5. SCHWARZIAN DERIVATIVE AND UNIVALENCE

5.1 Constant  $\sigma_3$ . Let A again be a simply connected domain with more than one boundary point. As a kind of opposite to the constant  $\sigma_2$  we define

 $\sigma_3 = \sup \{a \mid || S_f || \leq a \text{ implies } f \text{ univalent in } A\}.$ 

Note that the number a = 0 is always in the above set. In this definition, sup can be replaced by max, as can be shown by a standard normal family argument.

Nehari [12] proved that in a disc, the condition  $||S_f|| \leq 2$  implies the univalence of f, and Hille [5] showed that the bound 2 is best possible. In other words,  $\sigma_3 = 2$  for a disc.

A closer study of  $\sigma_3$  leads to the universal Teichmüller space and reveals an intrinsic significance of quasiconformal mappings in the theory of univalent functions. The gist is the following result.

THEOREM 5.1. The constant  $\sigma_3$  is positive if and only if A is bounded by a quasicircle.

*Proof*: The sufficiency of the condition was established by Ahlfors [1] who actually proved more: If A is bounded by a K-quasicircle, there is an  $\varepsilon > 0$  depending only on K, such that whenever  $|| S_f ||_A < \varepsilon$ , then f is univalent and can be continued to a quasiconformal mapping of the plane. In the proof, the extension of the given meromorphic f is explicitly constructed by means of a continuously differentiable quasiconformal reflection  $\varphi$  in  $\partial A$  with bounded  $| d\varphi | / | dz |$  (cf. 3.3).

The necessity was proved by Gehring [2]. His proof was in two steps. It was first shown, by aid of an example, that if A is not *b*-locally connected for any *b*, then  $\sigma_3 = 0$ . After this, the desired conclusion was drawn from the result we stated above as Lemma 3.2.

5.2 Universal Teichmüller space. Henceforth, we assume that the domain A is bounded by a quasicircle. Let Q(A) be the Banach space