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1° Any two points of the set $E \cap \mathrm{cl}\,U(z, r)$ can be joined by an arc lying in $E \cap \mathrm{cl}\,U(z, br)$.

2° Any two points of the set $E - U(z, r)$ can be joined by an arc lying in $E - U(z, r/b)$.

The following result has recently been proved by Gehring [2]:

LEMMA 3.2. *Let the set C contain at least two points and bound a simply connected domain A. If A is b-locally connected, then C is a $c(b)$-quasicircle, where $c(b)$ depends only on b.*

3.3 *Quasiconformal reflection.* Let C be a Jordan curve bounding the domains A and B. A sense-reversing K-quasiconformal mapping $\varphi\colon A \to B$ is a *K-quasiconformal reflection* in C if φ leaves every point of C invariant.

It is not difficult to prove that C admits a quasiconformal reflection if and only if C is a quasicircle. It follows that a quasiconformal mapping $f\colon A \to B$ between domains A and B bounded by quasicircles can be extended to a quasiconformal mapping of the plane. In fact, if φ and ψ are quasiconformal reflections in the boundaries ∂A and ∂B, such that φ is defined outside A and ψ in B, then $\psi \circ f \circ \varphi$ extends f quasiconformally.

A quasicircle always admits quasiconformal reflections which are continuously differentiable or even real-analytic. For a K-quasicircle passing through ∞, a reflection φ exists such that $|d\varphi(z)| / |dz|$ is bounded by a constant depending only on K.

For more details of the properties of quasicircles we refer to [10].

4. Deviation of a domain from a disc

4.1 *Schwarzian derivative.* Let f be a locally injective meromorphic function in a simply connected domain A. At finite points of A which are not poles of f, the *Schwarzian derivative* S_f of f is defined by

$$S_f = (f''/f')' - \frac{1}{2}(f''/f')^2,$$

and the definition is extended to ∞ and to the poles of f by means of inversion.

The Schwarzian derivative is holomorphic in A. Conversely, every function which is holomorphic in A is the Schwarzian of some f. The Schwarzian vanishes identically if and only if f is a Möbius transformation.

More generally, the Schwarzian determines a function up to a Möbius transformation.

Suppose the boundary of A consists of more than one point; then a conformal mapping h of A onto the unit disc exists. Through h a conformally invariant metric $\rho(z)\,|dz|$ is defined in A, by the rule $\rho(z)\,|dz| = (1-|w|^2)^{-1}\,|dw|$, $w = h(z)$. For functions φ holomorphic in A we introduce the norm

$$\|\varphi\|_A = \sup_{z\in A} |\varphi(z)|\,\rho(z)^{-2}.$$

The Schwarzian obeys the composition rule $S_{f\circ g} = (S_f \circ g)\,g'^2 + S_g$. We note certain of its immediate consequences. First, let f be meromorphic in A and $h\colon A \to B$ a conformal mapping. Then

$$(4.1)\qquad |S_f(z) - S_h(z)|\,\rho_A(z)^{-2} = |S_{f\circ h^{-1}}(\zeta)|\,\rho_B(\zeta)^{-2}, \quad \zeta = h(z).$$

It follows that $\|S_f - S_h\|_A = \|S_{f\circ h^{-1}}\|_B$. In particular,

$$(4.2)\qquad \|S_h\|_A = \|S_{h^{-1}}\|_B.$$

Secondly, let f and g be meromorphic in A and $h\colon G \to A$ a conformal mapping. Then

$$(4.3)\qquad \|S_{f\circ h} - S_{g\circ h}\|_G = \|S_f - S_g\|_A.$$

Finally, we remark that the norm of the Schwarzian is completely invariant under Möbius transformations: If f is meromorphic in A and g and h are Möbius transformations, then $\|S_{h\circ f\circ g}\|_{g^{-1}(A)} = \|S_f\|_A$.

4.2 *Constant* σ_1. We associate with the domain A the constant $\sigma_1 = \|S_f\|_A$, where f is a conformal map of A onto a disc. Here a disc means an ordinary disc or a half-plane. The number σ_1 is well defined, and equal to 0 if and only if A itself is a disc. It can be regarded as a measure of how much the domain A differs from a disc.

It is well known that $\sigma_1 \leqslant 6$ (Theorem of Kraus [6]). For the domain $A = \{z \mid 0 < \arg z < k\pi\}$, $1 \leqslant k \leqslant 2$, we have $\sigma_1 = 2(k^2-1)$. If follows that σ_1 can take any value in the closed interval [0, 6].

4.3 *Domains bounded by a quasicircle.* In some cases, information about the boundary of A makes it possible to improve the estimate $\sigma_1 \leqslant 6$.

THEOREM 4.1. *For a domain A bounded by a K-quasicircle,*

$$(4.4)\qquad \sigma_1 \leqslant 6\,\frac{K^2-1}{K^2+1}.$$

Proof: By Lemma 3.1, there exists a K^2-quasiconformal mapping w of the plane whose restriction to the upper half-plane H maps H conformally onto A. For the function $w \mid H$ the Krauss estimate can be improved:

$$\| S_{w|H} \|_H \leqslant 6 \frac{K^2 - 1}{K^2 + 1};$$

for the proof we refer to Kühnau [7], or to [8]. Hence (4.4) follows from (4.2).

4.4 *Domains with bounded boundary rotation.* Let A be a domain bounded by a continuously differentiable Jordan curve. The total variation of the direction angle of the boundary tangent under a complete circuit is called the boundary rotation of A. If the boundary is not so regular, boundary rotation is defined by means of approximations from inside.

Let f be a conformal mapping of the unit disc D onto a domain A with boundary rotation $k\pi$, $2 \leqslant k < \infty$. A real-valued function ψ with the properties

$$\int_0^{2\pi} d\psi(\theta) = 2, \int_0^{2\pi} |d\psi(\theta)| = k,$$

can be associated with f, such that

$$f'(z) = f'(0) \exp\left(-\int_0^{2\pi} \log(1 - ze^{-i\theta})\, d\psi(\theta)\right). \tag{4.5}$$

The domain A is convex if and only if $k = 2$. This is equivalent to ψ being an increasing function. A function f whose derivative admits the representation (4.5) is always univalent if the total variation of ψ is $\leqslant 4$.

Domains with bounded boundary rotation were introduced by Löwner and their basic properties established by Paatero [14].

Theorem 4.2. *For a domain A with boundary rotation $\leqslant k\pi$, $2 \leqslant k \leqslant 4$,*

$$\sigma_1 \leqslant \frac{2k + 4}{6 - k}. \tag{4.6}$$

The bound is sharp.

Proof: Let $f: D \to A$ be a conformal mapping, z_0 an arbitrary point of D, and h a conformal self-mapping of D, such that $h(0) = z_0$. Since $\rho_D(0) = 1$, it follows from (4.1) that

$$(4.7) \qquad | S_f(z_0) | \rho_D(z_0)^{-2} = | S_{f \circ h}(0) | .$$

Hence, (4.6) follows if we prove that $| S_f(0) | \leqslant (2k+4) / (6-k)$. Since we may replace f by the function $z \to cf(ze^{i\varphi})$, c complex, φ real, there is no loss of generality in assuming that $S_f(0) \geqslant 0$ and that $f'(0) = 1$. From the representation formula (4.5) we then deduce that

$$(4.8) \qquad S_f(0) = \int_0^{2\pi} \cos 2\theta \, d\psi(\theta) - \frac{1}{2}\left(\int_0^{2\pi} \cos\theta d\psi(\theta)\right)^2 + \frac{1}{2}\left(\int_0^{2\pi} \sin\theta d\psi(\theta)\right)^2 .$$

If $k = 2$, we have $d\psi(\theta) \geqslant 0$. In this case we get the inequality $\sigma_1 \leqslant 2$ for convex domains from (4.8) quite easily, just by use of Schwarz's inequality. Extremal functions can also be determined. These computations have been carried out in [9]. Nehari [13] proved the result $\sigma_1 \leqslant 2$ by means of variational methods.

If $2 < k \leqslant 4$, establishing (4.6) requires a more careful handling of formula (4.8). These computations will be published in a joint paper with O. Tammi.

Matti Lehtinen has let me know that for functions f whose derivative satisfies (4.5) with a ψ whose total variation is $\leqslant k$, $k \geqslant 4$, the sharp upper bound for $\| S_f \|$ is equal to $(k^2-4)/2$. The extremal functions are not univalent.

4.5 *Constant* σ_2. The domain constant

$$\sigma_2 = \sup \{ \| S_f \|_A \mid f \text{ univalent in } A \}$$

is in simple relation with σ_1 ([9]):

THEOREM 4.3. *In every domain* A, $\sigma_2 = \sigma_1 + 6$.

Proof: Let f be univalent in A and $h: D \to A$ conformal. By (4.3),

$$\| S_f \|_A = \| S_{f \circ h} - S_f \|_D \leqslant 6 + \| S_h \|_D = 6 + \sigma_1 .$$

In order to derive an estimate in the opposite direction, let an $\varepsilon > 0$ be given. In view of formula (4.7), we can choose $h: D \to A$ so that $| S_h(0) |$

$> \sigma_1 - \varepsilon$. If w is defined by $w(z) = z + e^{i\theta} / z$ and $f = w \circ h^{-1}$, then f is univalent in A and

$$\| S_f \|_A = \| S_w - S_h \|_D \geqslant | S_w(0) - S_h(0) | = | 6e^{i\theta} + S_h(0) | .$$

By choosing φ suitably we obtain $\| S_f \|_A > 6 + \sigma_1 - \varepsilon$.

5. Schwarzian derivative and univalence

5.1 *Constant* σ_3. Let A again be a simply connected domain with more than one boundary point. As a kind of opposite to the constant σ_2 we define

$$\sigma_3 = \sup \{a \mid \| S_f \| \leqslant a \text{ implies } f \text{ univalent in } A\}.$$

Note that the number $a = 0$ is always in the above set. In this definition, sup can be replaced by max, as can be shown by a standard normal family argument.

Nehari [12] proved that in a disc, the condition $\| S_f \| \leqslant 2$ implies the univalence of f, and Hille [5] showed that the bound 2 is best possible. In other words, $\sigma_3 = 2$ for a disc.

A closer study of σ_3 leads to the universal Teichmüller space and reveals an intrinsic significance of quasiconformal mappings in the theory of univalent functions. The gist is the following result.

Theorem 5.1. *The constant* σ_3 *is positive if and only if* A *is bounded by a quasicircle.*

Proof: The sufficiency of the condition was established by Ahlfors [1] who actually proved more: If A is bounded by a K-quasicircle, there is an $\varepsilon > 0$ depending only on K, such that whenever $\| S_f \|_A < \varepsilon$, then f is univalent and can be continued to a quasiconformal mapping of the plane. In the proof, the extension of the given meromorphic f is explicitly constructed by means of a continuously differentiable quasiconformal reflection φ in ∂A with bounded $| d\varphi | / | dz |$ (cf. 3.3).

The necessity was proved by Gehring [2]. His proof was in two steps. It was first shown, by aid of an example, that if A is not b-locally connected for any b, then $\sigma_3 = 0$. After this, the desired conclusion was drawn from the result we stated above as Lemma 3.2.

5.2 *Universal Teichmüller space.* Henceforth, we assume that the domain A is bounded by a quasicircle. Let $Q(A)$ be the Banach space