Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 24 (1978)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: UNIVALENT FUNCTIONS, SCHWARZIAN DERIVATIVES AND QUASICONFORMAL MAPPINGS

Autor: Lehto, Olli

Kapitel: 3. Quasicircles

DOI: https://doi.org/10.5169/seals-49701

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 24.01.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

3. Quasicircles

3.1 *Definition.* A Jordan curve is the image of a circle under a homeomorphism of the plane. If the homeomorphism can be taken to be a K-quasiconformal mapping, the Jordan curve is called a *K-quasicircle.*

For a later application, we need the following result.

Lemma 3.1. *A K-quasicircle is the image of the real axis under a quasiconformal mapping of the plane which is conformal in the upper half-plane and K^2-quasiconformal in the lower half-plane.*

Proof: Let C be a K-quasicircle. Then there is a K-quasiconformal mapping w of the plane which carries the real axis onto C. Let μ denote the complex dilatation of w. By the existence theorem for Beltrami equations, there is a quasiconformal self-mapping h of the upper half-plane with complex dilatation μ. If h is extended to the lower half-plane by reflection in the real axis, we obtain a K-quasiconformal mapping of the plane. Then $w \circ h^{-1}$ has the desired properties: by the uniqueness theorem for Beltrami equations, it is conformal in the upper half-plane, and as a composition of two K-quasiconformal mappings it is K^2-quasiconformal in the lower half-plane.

The notion of a quasicircle was introduced by Pfluger [15]; he arrived at these curves, which he called "kreisähnlich", in connection with a sewing problem for Riemann surfaces. Pfluger proved that a quasicircle, while always of zero area, need not be rectifiable. Later, Gehring and Väisälä [4] showed that the Hausdorff dimension of a quasicircle is always < 2 but can take any value λ, $1 \leqslant \lambda < 2$.

3.2 *Geometric characterization.* The first systematic study of quasicircles is Tienari's thesis [16]. His results were soon overshadowed by Ahlfors [1], who gave an amazingly simple geometric characterization of quasicircles: A Jordan curve passing through ∞ is a quasicircle if and only if for any of its three successive finite points z_1, z_2, z_3, the ratio $|z_1 - z_2| : |z_1 - z_3|$ is uniformly bounded.

The condition of Ahlfors can be modified in various ways. Let $U(z, r) = \{w \,\|\, w - z| < r\}$ and let $\mathrm{cl}U$ denote the closure of U. A set E of the extended plane is *b-locally connected* if the following two conditions hold for every finite z and every $r > 0$:

1° Any two points of the set $E \cap \mathrm{cl}U(z, r)$ can be joined by an arc lying in $E \cap \mathrm{cl}U(z, br)$.

2° Any two points of the set $E - U(z, r)$ can be joined by an arc lying in $E - U(z, r/b)$.

The following result has recently been proved by Gehring [2]:

LEMMA 3.2. *Let the set C contain at least two points and bound a simply connected domain A. If A is b-locally connected, then C is a $c(b)$-quasicircle, where $c(b)$ depends only on b.*

3.3 *Quasiconformal reflection.* Let C be a Jordan curve bounding the domains A and B. A sense-reversing K-quasiconformal mapping $\varphi: A \to B$ is a *K-quasiconformal reflection* in C if φ leaves every point of C invariant.

It is not difficult to prove that C admits a quasiconformal reflection if and only if C is a quasicircle. It follows that a quasiconformal mapping $f: A \to B$ between domains A and B bounded by quasicircles can be extended to a quasiconformal mapping of the plane. In fact, if φ and ψ are quasiconformal reflections in the boundaries ∂A and ∂B, such that φ is defined outside A and ψ in B, then $\psi \circ f \circ \varphi$ extends f quasiconformally.

A quasicircle always admits quasiconformal reflections which are continuously differentiable or even real-analytic. For a K-quasicircle passing through ∞, a reflection φ exists such that $|d\varphi(z)| / |dz|$ is bounded by a constant depending only on K.

For more details of the properties of quasicircles we refer to [10].

4. DEVIATION OF A DOMAIN FROM A DISC

4.1 *Schwarzian derivative.* Let f be a locally injective meromorphic function in a simply connected domain A. At finite points of A which are not poles of f, the *Schwarzian derivative* S_f of f is defined by

$$S_f = (f''/f')' - \frac{1}{2}(f''/f')^2,$$

and the definition is extended to ∞ and to the poles of f by means of inversion.

The Schwarzian derivative is holomorphic in A. Conversely, every function which is holomorphic in A is the Schwarzian of some f. The Schwarzian vanishes identically if and only if f is a Möbius transformation.