2. Quasiconformal mappings

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **24 (1978)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **28.04.2024**

UNIVALENT FUNCTIONS, SCHWARZIAN DERIVATIVES AND QUASICONFORMAL MAPPINGS[1]

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1. INTRODUCTION

Univalent functions have been a popular topic in complex analysis for over sixty years. It has also been known for a long time that there are interesting connections between univalence and the Schwarzian derivative. More recently, one has discovered in this interplay the important role of quasiconformal mappings which not only provide a tool but, somewhat surprisingly, are intrinsic in the problem of deducing univalence from the behavior of the Schwarzian. In this survey, we shall describe some recent developments in this area.

After defining plane quasiconformal mappings, we briefly discuss quasicircles in Section 3. These curves, introduced by Pfluger [15] in 1960, play a central role in this survey. Section 4 deals with the problem of measuring the deviation of a simply connected domain A from a disc D by means of the Schwarzian derivative of the conformal mapping function $f: A \to D$. The starting point in Section 5 is the remarkable result that in a simply connected domain, a small Schwarzian derivative implies univalence if and only if the boundary of the domain is a quasicircle. The sufficiency of this condition is due to Ahlfors [1], the necessity to Gehring [2]. This result gives rise to considering the universal Teichmüller space, and in this way various explicit estimations for certain domain constants can be derived ([9]).

2. QUASICONFORMAL MAPPINGS

2.1 *Module of a curve family.* Roughly speaking, quasiconformal mappings are homeomorphisms under which conformal invariants remain quasi-invariant. A precise definition can be given, for instance, in terms of the module of curve families. Let A be a domain in the plane and Γ a family

[1]) Communicated to an International Symposium on Analysis, held in honour of Professor Albert Pfluger, ETH Zürich, 1978.

of Jordan arcs or curves lying in A. Consider non-negative Borel functions ρ in A and denote by $P(\Gamma)$ the family of all such functions with the property $\int_\gamma \rho \, |dz| \geqslant 1$ for every locally rectifiable $\gamma \in \Gamma$. The greatest lower bound

$$M(\Gamma) = \inf_{\rho \in P(\Gamma)} \int_A \rho^2$$

is called the module of the family Γ.

A sense-preserving homeomorphism f of A onto another domain of the plane is a *K-quasiconformal mapping* if

$$M(\Gamma) / K \leqslant M\big(f(\Gamma)\big) \leqslant K\, M(\Gamma) \tag{2.1}$$

for every family Γ whose elements lie in A. The smallest possible K in (2.1) is called the maximal dilatation of f. A sense-preserving homeomorphism is conformal if and only if it is 1-quasiconformal.

2.2 *Beltrami equation.* Another way to characterize quasiconformality is as follows: A sense-preserving diffeomorphism is K-quasiconformal if it takes infinitesimal circles onto infinitesimal ellipses with a ratio of axes $\leqslant K$. A sense-preserving homeomorphism is K-quasiconformal if it is the limit of K-quasiconformal diffeomorphisms in the topology of locally uniform convergence.

A variant of this definition is based on the notion of L^2-derivatives. A continuous function is said to have L^2-derivatives in A if it is absolutely continuous on lines in A and if its partials, which then exist a.e. in A, are locally square integrable. By use of complex derivatives ∂ and $\bar{\partial}$, one more equivalent definition of quasiconformity is the following: A function f is a K-quasiconformal mapping of A if and only if f has L^2-derivatives in A and satisfies a Beltrami equation $\bar{\partial} f = \mu \, \partial f$ a.e. in A, where the function μ, the complex dilatation of f, is bounded in absolute value by $(K-1)/(K+1)$.

The existence theorem for Beltrami equations says that every function μ which is measurable in A and for which $\|\mu\|_\infty < 1$ agrees a.e. with the complex dilatation of a quasiconformal mapping of A. By the uniqueness theorem, complex dilatation determines a quasiconformal mapping up to conformal transformations.

For more details about the properties of quasiconformal mappings in the plane we refer to [11].