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ORIGINS OF THE COHOMOLOGY OF GROUPS [1]

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1. THE HISTORICAL QUESTIONS

This paper is a small essay on the history of contemporary mathematics. It will examine the questions: What were the origins of the theory of the cohomology of groups? What were the essential steps in the development of this theory? What were the effects of this development in related fields of mathematics? These questions seem appropriate to a conference in Zurich, because major steps in the development of this subject took place here at the Eidgenössische Technische Hochschule. These questions may also be typical of questions that can be put about the development of other parts of mathematics in the twentieth century. Here are some of these questions: How does the interchange of ideas between different fields of mathematics come about? Which ideas (or, which research papers) are of essential novelty or originality and which are derivative? Do some ideas arrive before their time, and so are neglected? What are the differences between mathematical developments seen beforehand, or seen after the fact—and is there not a third perspective, that of mathematical ideas as they are in process of development?

2. FUNDAMENTAL GROUP AND 2ND BETTI GROUP

On September 12, 1941 Heinz Hopf communicated to the *Commentarii Mathematicii Helvetici* his paper "Fundamentalgruppe und zweite Bettische Gruppe". This paper proved the

THEOREM. Each group G determines, by an algebraic process, a group G_1^* which is not generally zero. If G is the fundamental group of a complex K with second Betti group $B^2 = H_2(K, \mathbf{Z})$, and if S^2 is the spherical subgroup of B^2, then

$$B^2/S^2 \cong G_1^* \qquad (1)$$

[1]) Presented at the Colloquium on Topology and Algebra, April 1977, Zurich.

In this theorem, a homology class in B^2 belongs to the subgroup S^2 of spherical cycles when it can be represented by a continuous image of a 2-sphere. The algebraic construction process used in this theorem was the following: Represent the fundamental group G as $G \cong F/R$, where F is a free group and R a subgroup of F, form the subgroup $[F, R]$ generated by the commutators $frf^{-1}r^{-1}$ for $f \in F$ and $r \in R$ and the corresponding commutator subgroup $[F, F]$. Then the factor group

$$G_1^* = R \cap [F, F]/[F, R] \tag{2}$$

is independent of the choice of the representation $G = F/R$ of G as a quotient group of a free group. This G_1^* is the algebraic construction used in (1) above to measure the influence of the fundamental group on the second homology group. For example, if G is a free abelian group of rank p, then G_1^* is free abelian of rank $p(p-1)/2$, so this last integer is a lower bound for the second Betti number of a complex with fundamental group G. In general, as Hopf observed, this "lower bound" G_1^* cannot be improved; for every finitely presented group G one can readily construct a complex K with G as fundamental group and with $S^2 = 0$, so that $H_2(K, Z)$ is exactly G_1^*.

The essential originality in this theorem of Hopf resides in its use of a non-obvious purely group-theoretic construction (2) in order to express the dependence of one topological invariant (here B^2/S^2) upon another, the fundamental group. What had been known before this? It had long been known that the fundamental group G determined the 1-dimensional Betti group B^1 as the factor commutator group $B^1 = G/[G, G]$. This was a fact which had a direct geometric interpretation and involved only an evident—and evidently invariant—construction on G. Hopf's construction of G_1^* was much more subtle, and required a proof that the result is independent of the choice of the representation $G = F/R$. Actually, this group construction (2) had been known before—it is exactly the Schur multiplicator of the group G. This multiplicator had been introduced by Schur in his study of the projective representations of groups. Hopf, while a student in Berlin, had been an assistant to Schur, but his 1942 paper does not mention the connection with the multiplicator. Instead, his motivation seems to have come more from his earlier studies of the homology of Lie groups. As Eckmann pointed out to me, Hopf described this connection in his 1946 (written 1941) "Report on some new results in topology". He says of his theorem above that "The proofs rest on the idea that systems of curves which represent certain finite systems of elements of the fundamental group span surfaces in the complex, whose contribution to B^2 can be specified... in some cases, the surface is a torus, as in the case of the Pontrjagin product in a group manifold".

In his 1942 paper, Hopf did not mention a connection with the higher homotopy groups, but this connection soon played an important part.

Hurewicz introduced the higher homotopy groups in 1935. In 1936 he proved for an aspherical complex (one with all higher homotopy groups zero) that the fundamental group did determine all the Betti groups: This meant that two such complexes K and K' with isomorphic fundamental groups would have isomorphic Betti groups $B^n \cong B'^n$ in all dimensions n. In particular, it showed for a complex K with $S^2 = 0$ that B^2 would depend only on the fundamental group G. Hurewicz did not determine the fashion of this dependence, though according to Freudenthal [1946] he did raise this question in conversations. In effect, Hopf's paper provided the answer to the question of Hurewicz for $n = 2$.

Hopf's 1942 paper was the starting point for the cohomology and homology of groups; indeed this Hopf group G_1^* is simply our present second homology group $H_2(G, \mathbf{Z})$. This idea and this paper were indirectly the starting point for several other developments: Invariants of group presentations; cohomology of other algebraic systems; functors and duality; transfer and Galois cohomology; spectral sequences; resolutions; Eilenberg-Mac Lane spaces; derived functors and homological algebra; and other ideas as we will indicate below.

After the fact, we can view Hopf's paper as the first decisive step in the development of group cohomology and homological algebra. Beforehand, it appears differently, as a specific answer to a question implicit in the work of Hurewicz: Exactly how does the fundamental group affect the second Betti group? During the process, it was soon apparent from Hopf's paper that something exciting was going on. The review by Hassler Whitney, in *Math. Reviews*, Vol. 3 (1942), p. 316 says in its first paragraph:

"This paper is, in the reviewer's mind, one of the most important contributions to combinatorial topology in recent years. It gives far reaching results concerning the relations between the fundamental group, the first and second homology and cohomology groups, and the products between these groups, with beautiful and simple methods. The work is based on some new constructions in groups which are undoubtedly of real significance by themselves. The paper is in three main parts: the group theory; determination of the second homology group B^2 (all groups are with integer coefficients) modulo the group S^2 of "spherical homology classes" [see below]; and the determination of the products for dimensions not greater than 2 (omitting considerations of torsion). In each case, the formulas are in terms of the fundamental group G, and pure group-theoretic constructions, but with geometric meanings."

3. Homology and Cohomology of Groups

As Whitney's review does suggest, Hopf's paper had immediate influence. His description of the second integral homology group of a group G was followed by four independent studies, two of which described the higher homology groups $H_n(G, \mathbf{Z})$ and two the higher cohomology group $H^n(G, A)$ for an abelian group A or, more generally, for a G-module A. Each of these papers explicitly recognizes the starting point provided by the paper of Hopf. In chronological order, these four studies are as follows:

Eilenberg and Mac Lane [1942] had been applying methods of group extensions to the universal coefficient theorem in cohomology, so they knew the group Ext (G, A) of all abelian extensions of the abelian group A by the abelian group G. They knew that a representation of G as F/R, with F and R free abelian, would give an exact sequence

$$0 \to \hom(G, A) \to \hom(F, A) \to \hom(R, A) \to \operatorname{Ext}(G, A) \to 0$$

(though they expressed this fact differently, writing Ext (G, A) as a suitable quotient of hom (R, A)). Moreover, they had heard of the Schur multiplicator through Mac Lane's work on class field theory. Furthermore, Eilenberg was very familiar with homotopy groups. Hence, as soon as they saw the Hopf 1942 paper, they decided that more group extensions must be hidden in Hopf's G_1^*, and they set about to find out how.

On April 7, 1943 Eilenberg and Mac Lane submitted to the *Proceedings* of the National Academy of Sciences an announcement "Relations between homology and homotopy groups". Given a group G, they constructed a chain complex $K(G)$, whose second homology group is exactly Hopf's group G_1^*. Their complex $K(G)$—now called the Eilenberg-Mac Lane complex $K(G, 1)$—had as generators in dimension n the cells $[x_1, ..., x_n]$ for entries $x_i \in G$, with boundary

$$\partial [x_1, \ldots, x_n] = [x_2, \ldots, x_n] + \sum_{i=1}^{n-1} (-1)^i [x_1, \ldots, x_i x_{i+1}, \ldots, x_n] + (-1)^n [x_1, \ldots, x_{n-1}] .$$

(If we use these cells to generate a free G-module and add the operator x_1 to the first boundary terms, this is just the bar resolution.) For any dimension n, the cohomology groups $H(K(G), A)$ with coefficients in an abelian group A were called the cohomology group of G with coefficients A. The essential topological result reads

THEOREM. If a space X is arcwise connected and has vanishing homotopy groups

$$\Pi_n(X) = 0 \qquad 1 < n < r$$

then the n-dimensional cohomology of X is given by

$$H^n(X, A) \cong H^n(\Pi_1(X), A), \qquad n < r,$$
$$\Lambda^r(X, A) \cong H^r(\Pi_1(X), A), \qquad n = r.$$

Here $\Lambda^r(X, A)$ is the subgroup of the r dimensional cohomology group H^r consisting of those cohomology classes which annihilate the spherical subgroup $S^r(X)$—consisting of those integral homology classes which can be represented by continuous images of spheres (as in the case of S^2 in Hopf's theorem for a polyhedron K).

In this paper there was also a corresponding theorem for the homology of X. It was formulated for the singular homology of an arbitrary space X, rather than for a complex, as in the work of Hopf. This is essentially a technical change, made possible by the fact that Eilenberg [1944] in the meantime had carried out the definitive formulation of singular homology. The essential fact was the same: The algebraic formulation of the influence of Π_1. In the simplest case: For an arcwise connected space X which is aspherical ($\Pi_n(X) = 0$ for all $n > 1$), the homology and cohomology of X depend only on the fundamental group $\Pi_1(X)$ and can be expressed algebraically as the homology and cohomology of the group $\Pi_1(X)$.

This paper of Eilenberg-Mac Lane also establishes briefly the corresponding result for an arcwise connected space X with exactly one non-vanishing homotopy group $\Pi_n(X)$. (An "Eilenberg-Mac Lane space") again by way of a suitable chain complex $K(\Pi_n(X), n)$ which represented, in algebraic form, a "minimal" singular complex of such a space X.

The next paper chronologically was Hopf's paper (communicated April 1, 1944 to *Commentarii*) "Uber die Bettische Gruppen, die zu einer beliebige Gruppe gehören". This paper describes the homology groups of a group G with coefficients in a G-module J. First form an exact sequence (Hopf didn't call it that or write it so!)

$$0 \leftarrow J \leftarrow X^0 \leftarrow X^1 \leftarrow \ldots \leftarrow X^n \leftarrow \qquad (3)$$

of free G-modules, that is, of free modules over the integral group ring $P = Z[G]$. Regard the group $\mathbf{Z}$ of integers as a G-module with trivial action. Then the homology of the complex

$$\mathbf{Z} \otimes_G X^0 \leftarrow \mathbf{Z} \otimes_G X^1 \leftarrow \ldots \leftarrow \mathbf{Z} \otimes_G X^n \leftarrow$$

is independent of the choice of the exact sequence (3). Its n^{th} homology group is called the n^{th} Betti group $H_n(G, J)$. Moreover, Hopf proves that these algebraically defined groups are the homology groups of an arcwise connected aspherical space with fundamental group G, exactly as in the case above.

In formulating these facts we have changed Hopf's technique slightly. He didn't speak of exact sequences, because he hadn't yet had "the word" (which was invented about this time by Eilenberg and Steenrod for their axiomatic treatment of homology). He didn't explicitly use the tensor product $\mathbf{Z} \otimes_G X$ but instead used a quotient of X, which amounted to taking the augmentation map $\mathbf{Z}(G) \to \mathbf{Z}$, the corresponding short exact sequence $I(G) \rightarrowtail \mathbf{Z}(G) \to \mathbf{Z}$ and tensoring it with X. These are wholly minor differences. The essential fact is that Hopf had a clear formulation of the use of a *free* resolution and of the comparison theorem for two such resolutions (ideas not present in the Eilenberg-Mac Lane theorem cited earlier). Moreover his argument for his result replaced the aspherical space X by its universal covering space. Hence his use of different resolutions is clearly derived from the topological fact that *different* subdivisions of the same acyclic space (the universal covering space) will yield the same equivariant cohomology.

The third paper is by H. Freudenthal "Der Einfluss der Fundamental Gruppe auf die Bettischen Gruppen", published in the *Annals of Mathematics* in April 1946 and submitted there some time before July 29, 1945 (probably smuggled out of the Netherlands during the war). The paper was based on the *first* Hopf paper; because of the difficulty of communication during the war, its author did not know of the work of Eilenberg-Mac Lane, nor of the 1944 paper by Hopf discussed just above. Freudenthal's paper again uses free resolutions to define the homology and cohomology groups of G, and establishes essentially the same theorems relating these groups to the groups of an arcwise connected space aspherical in low dimensions. His use of free resolutions is again clearly a reflection of the properties of universal covering spaces.

The fourth paper, by Beno Eckmann "Der Cohomologie Ring einer beliebigen Gruppe", was communicated to the Commentarii on December 4, 1945. At that time Eckmann knew of both papers of Hopf, but did not know the papers of Eilenberg-Mac Lane or of Freudenthal. Given the group G and a ring J (with unit) his paper describes the cohomology ring $H^*(G, J)$ of G with coefficients in J. (In present terminology, this is a graded ring composed of various homology groups $H^n(G, J)$, each in its appropriate dimension n.) These cohomology groups are described by suitable cocycles and the ring structure is given by a suitable product, modeled after the Čech-Whitney cup product in topology. The main theorem again asserts that an arcwise connected space X with fundamental group G and aspherical in dimensions less the n has its cohomology ring in these dimension given by $H^*(G, J)$.

Eckmann also describes his cohomology group $H^n(G, J)$ as the cohomology of a chain complex K_G. This complex K_G is identical to the Eilenberg-Mac Lane complex $K(G, 1)$—but differently described. For Eilenberg-Mac Lane the n-cells of $K(G, 1)$ are the n-tuples $[x_1, ..., x_n]$ of elements $x_i \in G$. For Eckmann they are n-tuples $[y_1, ..., y_n]$; the translation is $y_i = x_1 \dots x_i$ for $i = 1, ..., n$.

The ring structure, clearly formulated in Eckmann's paper, had been noted in the other three papers—as a cup product structure in Eilenberg-Mac Lane and as a (intersection) structure in Hopf and Freudenthal.

Thus we have four substantially independent discoveries of the same facts: The algebraic definition of the n-dimensional homology (or cohomology) of a group G and its identification with the homology (or cohomology) of a suitably aspherical space with fundamental group G. All four papers are based on (and inspired by) the original paper of Hopf for $n = 2$. The fact that there were as many as four substantially independent discoveries is undoubtedly due to the sharply limited international communication during wartime. This unintended experiment does go to show that the first Hopf paper was a breakthrough, recognized as such. Because of its structure, more development was possible—and was sure to be carried out.

Such a breakthrough itself must depend on previous ideas and developments. In this case the breakthrough involved a continuation of ideas both from algebra and from topology; we now turn to examine these.

4. The Background in Abstract Algebra

Algebra treated by the use of axioms probably began in the late 19th century in the work of Dedekind, followed by E. H. Moore, Steinitz and others. However, modern or "abstract" algebra is a newer subject; it involves the use of axioms *and* conceptual methods to get a deeper understanding of disparate algebraic phenomena. As I have indicated elsewhere in a paper [1978] on the history of this subject, abstract algebra in this sense came into being in 1921, with a paper by Emmy Noether on "Ideal theorie in Ringbereichen". This was the first paper in which "Ring" was used in its modern axiomatic sense, though the word "number ring" had been used in Hilbert's 1898 *Zahlbericht*, while Fraenkel in 1916 had made partial attempts at axiomatics for rings. More important in this paper of Noether's was the clear recognition that some arithmetic theorems known for special rings of algebraic integers or of polynomials could be formulated and proved better under general conditions—avoiding needless computational complexities and bringing out the conceptual structures involved.

This paper of Noether's appears to have quickly stimulated many other studies in abstract algebra—both her own studies and those of her colleagues, collaborators, and pupils. In ten years, this provided the full background of abstract algebra, as formulated in van der Waerden's book *Moderne Algebra* (Band I, 1930; Band II, 1931). The abstract spirit was clearly there, though some of the central notions do not yet have due emphasis. For example, the notion of a (left) module over a ring, so important for the cohomology of groups, came in a bit indirectly under the titles "linearformenmoduln" and "groups with operators".

Related to our topic is the study of group extensions, which was stimulating by this emphasis on abstract algebra. The topic had come up before, at least implicitly, in I. Schur's study of the projective representation of a group and hence of the multiplicator. Group extension themselves were codified by Schreier [1926]. A group E with abelian normal subgroup A and quotient group G, in other words a short exact sequence

$$1 \to A \xrightarrow{i} E \xrightarrow{p} G \to 1\,, \tag{1}$$

is an *extension* of A by G. In such an extension, conjugation in E makes A a left $E/A = G$ module. If one chooses to each $x \in G$ a representative $u(x) \in E$ with $pu(x) = X$, the product of two such representatives has the form

$$u(x)u(y) = f(x, y)u(xy) \tag{2}$$

where the factor $f(x, y) \in A$ must satisfy an identity representing the associative law for the triple product $u(x)u(y)u(z)$; this f was called a "factor set". The extension E is then completely determined by G, the G-module A, and this factor set f. It turns out from the associative law that f is exactly a two dimensional cocycle for G, and that the set of all extensions E of the given G module A by G is exactly the two dimensional cohomology group $H^2(G, A)$. Hence this group, as well as the one dimension cohomology group $H^1(G, A)$, was well known in the 1930's. This made it possible for Eilenberg-Mac Lane and Eckmann to recognize in their papers cited above that the general cohomology of a group includes for dimension 2 the known case of group extensions.

In this description of group extensions by factor sets, the binary operation (of multiplication or perhaps addition) which makes $H^2(G, A)$ a group is given by the multiplication of two factor sets f, f' to form a new factor set

$$f(x, y)f'(x, y).$$

In his studies of group extension [1934], Baer had raised and answered the question of finding an invariant way of describing this multiplication of two extensions (1) and (1′)—a description independent of the choice of representatives and now called the "Baer product" of extensions. He likewise had considered extensions of a *non-abelian* group A by a group G, and had observed that such an extension, realizing given operators of G on A, are not always possible. Indeed, there is a certain obstruction to forming such an extension, and this obstruction is a three-dimensional cohomology class of $H^3(G, Z)$ where Z is the center of A. This obstruction was identified in this way by Eilenberg-Mac Lane in 1947, and was a central element in the development of the cohomology of groups as an independent subject, not necessarily tied to the motivating topological examples.

5. The Background in Class Field Theory

In the early 20th century, linear algebra was an Anglo-American subject. Hamilton's discovery of quaternions and C. S. Peirce's utilization of idempotents had started the subject off. In 1905 Wedderburn had proved that any finite division algebra was commutative; one year later he proved his structure theorems. In a sense, they reduced the search for all finite dimensional linear algebras to that for all division algebras. Dickson found many, by the study of cyclic algebras. These were algebras of order n^2 over a field K constructed from a cyclic extension N of K of degree n. The crucial ideas of this line of thought were recorded in Dickson's [1923] book "*Algebras and Their Arithmetics*". Its second edition was translated into German in 1927. This translation immediately attracted attention in Germany.

In the early 20th century, algebraic number theory was a Germanic subject. Hilbert's 1898 *Zahlbericht* (where he introduced the term "number ring") had led him to his study of relative quadratic fields. He conjectured that results he found could extend to a general class field theory. This was done 1920-1933 by Takagi, Feutwangler, Artin, Hasse, Chevalley and others. Indeed the development was one of the major driving forces behind the development of abstract algebra in Germany. Part of it dealt with local class field theory (that is, over a field k of p-adic numbers). There one wished in particular to determine over a local field k all the central simple algebras. These (as Brauer, Hasse, and Noether observed in 1932) could all be described as crossed product algebras, as follows. Take a finite normal extension field N of k, with Galois group G. In the vector space E over N of dimension the degree $n = [N: k]$ and basis the n elements u_x for $x \in G$ introduce a product by the rule

$$u_x u_y = f(x, y)\, u_{xy}$$

where f is a factor set of G in the multiplicative group N^* of N. With this product, E becomes a central simple algebra over k, and all central simple algebras over a local field, k, can be so represented. On the one hand, this generalizes Dickson's cyclic algebras from his case when the group G is a cyclic one. On the other hand, it describes the possible central simple algebras by factor sets, which in turn are just two-dimensional cocycles of G in N. So here again it is that cohomology enters algebra. In my own case, this is where I first learned of the two-dimensional cohomology group $H^2(G, N^*)$. A long study with Schilling attempting to extend class field theory to non-abelian extensions involved difficulties—and I recall thinking at the time that these difficulties came up because there were no *three*-dimensional factor sets available. Without a topological motivation, Schilling and Mac Lane did not discover the three-dimensional cohomology group $H^3(G, N^*)$. Teichmüller [1940] in a closely related problem about central simple algebras, *did* describe three-dimensional cohomology groups. He did nothing with them (probably because he found the study of complex moduli more fruitful, or perhaps because he was distracted by the war).

Others (Eilenberg-Mac Lane and Eckmann) dutifully cited Teichmüller's results—but it seems unlikely that those results really affected the development.

6. Betti Numbers or Homology Groups

The period 1927-1937 saw an extensive algebraization of combinatorial topology; this process was an essential prerequisite to the cohomology of groups. Before 1927, topology really was combinatorial: a chain in a complex was a string of simplices, each perhaps affected with a multiplicity (a coefficient), and the algebraic manipulation of chains was something auxiliary to their geometric meaning. This is undoubtedly as it must be, at the start; only later can it develop that geometric results follow from long algebraic computations which are not geometrically visible, step by step.

Combinatorial topology, following Poincaré, measured the connectivity of a polyhedron by its Betti numbers and torsion coefficients in each dimension, calculated as they were from chains and their boundaries. Between 1927 and 1934, the style changed completely; now the connectivity was measured by the homology groups, one in each dimension; the invariants of these abelian groups gave the previous Betti numbers and torsion coefficients. It is fascinating to trace this change, as best we now can. I can find no mention of homology *groups* before 1927; for example, the famous 1915 and 1926 papers of Alexander proved the invariance of the Betti *numbers* of a complex, not the invariance of the homology groups. Veblen's *Analysis Situs* (first edition, 1921; second edition, 1931) is all phrased in terms of incidence matrices and Betti numbers, except for one brief section in the back of the book where it is noted that the homology classes module p form a group.

Then in 1927 Vietoris studied the homology of spaces which were not necessarily polyhedra, so that the homotopy groups were not necessarily finitely generated—so of course he (had to) use homology groups. W. Mayer, with references to courses by Vietoris, used homology groups in a 1929 paper on "Abstract Topology" (submitted, November 1927). Heinz Hopf reviewed the paper in the *Jahrbuch*. In his review he notes, evidently with some surprise, that the paper used "group-theoretic methods". E. R. van Kampen's Dutch thesis "Die Combinatorische Topologie und die Dualititssatz", Den Haag 1929, formulates these ideas by homology *groups*. An influential article by Van der Waerden in 1930 summarized the state of topology: he used homology groups. Alexandroff (whose 1928 papers about compacta were all in terms of Betti numbers) used homology groups in his 1932 monograph on topology—but Alexandroff's review in the Jahrbuch of the 1927 paper by Vietoris doesn't even notice the use of groups.

A folk tale has it that homology groups first appeared in Göttingen. In the period 1926-1932 A. D. Alexandroff and Heinz Hopf frequently visited there; I heard Alexandroff lecture there on topology in 1932. At one time, perhaps in 1926, they were studying with some difficulty Lefschetz's proof of his fixed point theorem. They discussed it with Emmy Noether, who pointed out that the proof could be better understood by replacing the Betti numbers with the corresponding homology groups and using the trace of a suitable endomorphism of these groups. Other versions of the folk tale have it that Emmy simply observed that Betti numbers and torsion coefficients should be viewed as the standard invariants of a suitable abelian group, which should be the proper tool for the conceptual formulation of homological connectivity. It is not now clear whether or not this was the first use of the homology *groups*. At any rate, it is the case that these groups appear as such in the small 1932 book in which Alexandroff recorded his Göttingen lectures, while the 1935 book of Alexandroff and Hopf gives credit in the Preface to the advice of Emmy Noether.

In this case, it is difficult to identify a first use. It seems most likely that many topologists independently came to use homology groups rather than Betti numbers—and that this easy transition, much in keeping with the growth of abstract algebra, was not noted in any way as a special event. Only after the fact do we note a change—the development of mathematics in hindsight is seen under a very different perspective than at the time.

The use of homology groups was but a small part of the algebraization of topology. Another vital step clearly related to our story was the introduction of cohomology groups and the (cup product) cohomology ring. Previously one had used intersections for chains on manifolds. In 1935 it appeared that these intersections could be dualized to cup products of cochains—and that in this form the products would hold not just for manifolds, but for any polyhedra. This situation was recognized independently by Alexander, by Čech, Kolmogoroff and by Whitney; all in reports which they (except for Čech) planned to give at the 1935 conference on topology in Moscow. Whitney ultimately replaced his talk by one on another topic, but formulated his results on cup products in his decisive 1938 paper.

These observations about groups and homology may suffice to understand the trend 1927-1938 toward a thoroughgoing algebraic formulation of homology.

7. The Background in Homotopy

Group theory, in fact had been present in combinatorial topology from the beginning, in the study of the fundamental group (the Poincaré group) of a space or in particular of a manifold. The fundamental group Π_1 for a polyhedron P naturally comes with a presentation of the form $\Pi_1 = F/R$, where F is a suitable free group generated by circuits in the one-skeleton of P, while its subgroup R is described from the 2-cells of P. Hence *this* sort of presentation was ready at hand for Hopf's study of the influence of the fundamental group—and his paper does make reference to the work of Reidemeister, one of the German topologists concerned with the fundamental group.

The introduction of the higher homotopy group was more recent. At the 1932 International Congress of Mathematicians in Zurich, E. Čech had described our present two-dimensional homotopy group in a very brief note. He wrote no further on the subject. Folklore has it that other topologists at the conference discouraged him from further work, pointing out that his Π_2 was an abelian group, while all the experience with Π_1 indicated that what was wanted was a non-abelian group. Hence the real credit for the higher homotopy groups goes to W. Hurewicz, who introduced them in several brief notes in 1935-36, together with proofs of several of their properties—enough to show that these higher homotopy groups *did* have utility in topology. In particular, his 1936 theorem *that* the homology groups of an aspherical polyhedron are determined by the fundamental group of that polyhedron is the exact starting point of our subject.

Other developments at this time emphasized the importance of homotopy —Hopf's discovery [1931] of the essential maps of S^3 on S^2, and the work of Whitehead on combinatorial homotopy. It was clearly the right time to investigate the relation between homotopy and homology.

8. The Cohomology of Groups

Once launched by topology, the higher dimensional cohomology groups of a group took on a life of their own. Eilenberg-Mac Lane and Mac Lane separately examined properties of the group $H^n(G, A)$ for a general G-module A. They found (from the study of Baer) the purely group-theoretical interpretation of $H^3(G, A)$ by obstructions—but an equally useful interpretation for higher n, long sought for, is still missing (and may even not be there!). Eckmann introduced G-finite cohomology groups (1947) and showed their connection with the Hopf-Freudenthal theory of the *ends* of a group. Eckmann's work, and the paper of Eilenberg-Mac Lane on complexes with operators, again emphasized the connection of cohomology groups of groups with covering spaces. There was a systematic presentation of the subject in the Cartan seminar of 1950/51, entitled "Cohomologie des groupes, suite spectrale, faisceaux". In this seminar Eilenberg first described the cohomology groups axiomatically, and then proved their existence. Subsequent exposés by Cartan emphasized the calculation of the cohomology by free resolutions complete with an abstract version of the comparison theorem. A decisive example of the effective use of such resolutions is the calculation of the cohomology of a cyclic group—carried out here in exposé 3. (I am sensitive to the advantage of using resolutions for this purpose, because in 1948 I had calculated the cohomology of cyclic groups directly from the bar resolution *without* the general comparison theorem—the direct method worked but was much more cumbersome.) Subsequent exposés made a number of applications—to the Brauer group, the Wedderburn theorem, the theorem of Maschke on complete reducibility of linear representations of a finite group, and P. A. Smith's theorem.

Further applications to pure group theory have been limited. One small but striking one is the homology proof by Gaschutz [1966]:

THEOREM. A finite non-abelian p-group has an automorphism of p^{th} power order which is not an inner automorphism.

This conference in Zurich has exhibited more examples of the use of homology in group theory.

9. Spectral Sequences

The results stimulated by group cohomology were not confined just to group theory. For example, the problem of computing the cohomology groups $H^n(G, A)$ for the case when G itself is a group extension (say, cyclic by cyclic) immediately leads to the study of a spectral sequence. Specifically, if

$$1 \to K \to G \to Q \to 1 \qquad (1)$$

is a short exact sequence of (multiplicative) groups and A is a left G module there is a spectral sequence E_r^{pq} with

$$E_2^{pq} \cong H^p(Q, H^q(K, M)) \tag{2}$$

converging to the graded group associated with a filtration of the cohomology $E^{p+q}(G, M)$. In (2), the cohomology $H^q(K, M)$ of the subgroup K is suitably interpreted as a Q-module, so that the outside cohomology is defined. The essential portions of such a spectral sequence were discovered by R. Lyndon in his 1946 Harvard thesis, at about the same time that Leray was formulating the general notion of a spectral sequence. Lyndon did use his formulation for computation. Some years later [1953], Hochschild and Serre formulated a spectral sequence like that of (2) in the conventional language, so such a sequence is usually called a Hochschild-Serre spectral sequence. (There are actually several different constructions of such a sequence, and some residual uncertainty as to whether these constructions all yield the same spectral sequence). The essential observation is that computing cohomology or homology in a fiber situation like that of (1) inevitably leads to the spectral sequence technology—whether the fiber situation is group theoretic, as with the exact sequence (1), or a fiber space, as in the case so effectively exploited by Serre in topology.

10. Transfer

The operation of *transfer* was well known in group theory, beginning with Burnside's work on monomial representations. If H is a subgroup of index n in G, the transfer from G to H is a homomorphism.

$$t : G/[G, G] \to H/[H, H] \tag{1}$$

between the factor-commutator groups. To define it, choose representatives $x_1, \ldots, x_n$ of the right cosets of H in G, so that $G = \cup Hx_i$ and write $\rho(x)$ for the representative x_i of the coset Hx. Then t is

$$t(g) = \prod_{i=1}^{n} (x_i g)\,[\rho(x_i g)]^{-1} \tag{2}$$

This map t is independent of the choice of the set of representatives $x_i, \ldots, x_n$.

Since the factor commutator group $G/[G, G]$ in (1) is simply the 1-dimensional homology group $H_1(G, \mathbf{Z})$, the transfer can be regarded as a map in homology.

$$t : H_1(G, \mathbf{Z}) \to H_1(H, \mathbf{Z})$$

In 1953 Eckmann extended this map to apply in all dimensions, both in homology and cohomology. Using the standard homogeneous complexes

$B(G)$ and $B(H)$ for the groups G and H, he defined a cochain transformation t for any G-module A and any cochain f by

$$(tf)(g_0, \ldots, g_p) = \sum_{i-1}^{n} x_j^{-1} f(x_j g_0 (\rho(x_j g_0))^{-1}, \ldots, x_j g_n (\rho(x_j g_n)) - 1)$$

This map, up to chain homology, is again independent of the choice of the representatives x_j, so yields a homomorphism

$$t : H^p(H, A) \to H^p(G, A) .$$

On the other hand, each cochain of G over A automatically restricts to a cochain of H over A; this process defines the *restriction* map

$$r : H^p(G, A) \to H^p(H, A) .$$

Eckmann proved that the composite tr of these maps is the endomorphism given by multiplication by n in $H^p(G, A)$: He made a variety of applications. The notion of transfer was also used by Artin and Tate (see below) in class field theory.

The discovery of the homology of a group had the feature that it exhibited a "non-obvious" construction on groups; in much the same way, the discovery of transfer produced a non-obvious homomorphism between cohomology groups. Thus it is that recently Kahn and Priddy have been able to construct the transfer homeomorphism for the generalized cohomology of an n-fold covering $\Pi : E \to B$. This transfer applies to the cohomology with coefficients in any strict Ω-spectrum; when applied to the Eilenberg-Mac Lane spectrum $K(\Pi, n)$, the generalized cohomology is ordinary cohomology and the transfer agrees with the classical one. Using this transfer, they prove a conjecture of Mahowald and Whitehead about a "canonical map" of the n-fold suspension $\Sigma^n \mathbf{R}P^{n-1}$ of the real projective $n - 1$ space into the n sphere. This map λ is the adjoint of the map

$$\mathbf{R}P^{n-1} \to O_n \to \Omega^n S^n .$$

Here the first arrow takes a line through the origin in $\mathbf{R}^n$ into the reflection in the plane perpendicular to that line; while the second arrow represents each element of O^n as a map of $(\mathbf{R}_n \cup \infty, \infty)$ into itself, and hence as an element of the n^{th} loop space of S^n.

The result of Kan and Priddy is that λ is an epimorphism of 2-primary components in stable homotopy.

11. Class Field Theory

Some of the origins of the cohomology of groups—specifically, the factor sets for crossed product algebras—came from class field theory. Hence it is not surprising that one of the principle uses of this cohomology lies back in class field theory. Possibilities of this sort were in the minds of Eilenberg and Mac Lane when they wrote a paper applying cohomology of groups along the lines of the earlier Teichmüller work [1940] on 3-cocycles. Mac Lane also recalls that Artin (about 1948) pointed out in conversations that the cohomology of groups should have use in class field theory. Hochschild [1950] and Hochschild and Nakayama [1952] showed how the Brauer group arguments of class field theory could be replaced by cohomological arguments. In 1952, Tate proved that the homology and cohomology groups for a finite group G could be suitably combined in a single long exact sequence. He used this sequence, together with properties of transfer and restriction, to give an elegant reformulation of class field theory. It is still today one of the effective approaches to this subject—as presented, for example, in the recent book of Iyanaga and Iyanaga [1975].

12. Homological Algebra

The discovery of the cohomology of groups was an essential part of the development of homological algebra. This subject, as organized by Cartan and Eilenberg, provides a unified way of accounting for a variety of new functors, starting with the cohomology of groups. Such are:

$H^n(G, A)$, the cohomology of a group G, with coefficients in a left G-module A;

$H_n(G, A)$, the homology of a group G, with coefficients in a right G module A;

$H^n(\Lambda, A)$, the (Hochschild) cohomology of an algebra Λ, with coefficients in a Λ-bimodule A;

$H^n(g, C)$, the cohomology of the Lie algebra g, with coefficient in a g-module C;

Ext (A, B), the group of abelian group extensions of the abelian group B by the abelian group A;

Tor (A, B), the torsion product of the abelian groups A and B.

The first three functors (and others like them) all arose from our immediate subject, the cohomology of groups. The functor Ext is related since it describes a group of group extensions, but it enters our story more directly by its role in the universal coefficient theorem for homology; as found by Eilenberg-Mac Lane on the basis of a problem of Steenrod about regular cycles in metric spaces. Finally the Tor functor also came from the universal coefficient theorem in homology—and the functor Tor (without the name) first appears in connection with the universal coefficient theorem in a 1935 paper by Čech.

The decisive idea of homological algebra was the recognition that all these functors—as well as the higher Ext^n and $\mathrm{Tor}_n(A, B)$ for modules A and B over a ring R—could be described uniformly as the n^{th} "derived" functors of certain basic functors. Here the definition of derived functor rests on the notion of a projective resolution, which comes directly from the ideas of Hopf and Freudenthal on the homology of a group. For example, in this case, one regards the additive group $\mathbf{Z}$ of integers as a trivial $\mathbf{Z}$-module, forms an exact sequence.

$$\mathbf{Z} \leftarrow X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \dots$$

of projective left G-modules, tensors the result with A

$$A \otimes_G X_0 \leftarrow A \otimes_G X_1 \leftarrow A \otimes_G X_2 \leftarrow \dots$$

and calculates the homology of this complex in dimension n to obtain $H_n(G, A)$. For these homology groups, this is exactly the procedure used in Hopf's second paper to describe the Betti groups which belong to the group G—except that, as already noted, he did not have the tensor product of G modules at hand. He used only the trivial G module $A = \mathbf{Z}$, so he could describe our tensor product $\mathbf{Z} \otimes_G X$ as the quotient X/X_0, where X_0 is the submodule of X generated by all the finite sums $\Sigma\beta_i x_i$ with x_i in X, β_i in the group ring $\mathbf{Z}(G)$ and with augmentation $\alpha(\Sigma\beta_i) = 0$. In exact sequence terminology, this amounted to using the augmentation α to form a short exact sequence

$$I(G) \rightarrowtail \mathbf{Z}(G) \xrightarrow{\alpha} \mathbf{Z},$$

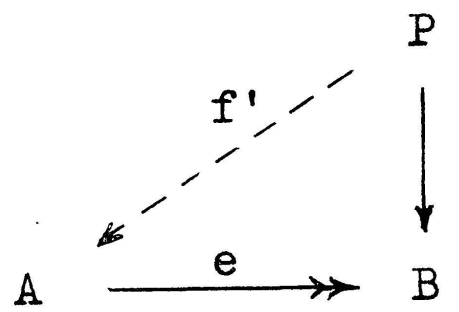
forming from this the right exact sequence

$$I(G) \otimes_G X \to \mathbf{Z}(G) \otimes_G X \cong X \to Z \otimes_G X \to 0$$

and hence getting $\mathbf{Z} \otimes_G X$ as the stated quotient of X. For us, it is easier now to use $\otimes_G$ directly—but Hopf's treatment shows that the ideas could still work without this explicit concept.

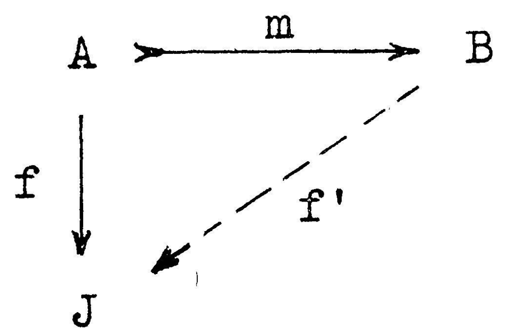
As observed, the strength of homological algebra lay in using the *same* method of resolution to describe derived functors of arbitrary additive functors—and this use of resolutions, together with the comparison theorem for different resolutions, came straight from the geometric properties of covering spaces as used by Hopf in his original construction. The other surprising aspect was the fact that homological algebra, formulated in this generality, had extensive applications to ring theory, especially through the consideration of homological dimension. It turned out that resolutions had really appeared before: In Hilbert's proof of his theorem on syzygies!

Actually, the complete theory of derived functors depends on the use of both projective and injective resolutions. A module P over the ring R is *projective* if every morphism f from P to the codomain of an epimorphism can be lifted to the domain as in the diagram



in other words if B is the codomain of an epimorphism e, each $f : P \to B$ factors as $f = ef'$ for some f'. In particular, a free R-module is evidently projective, so there are plenty of projective modules; in particular, every module is a quotient of a projective module.

The dual notion is that of an injective module J. A left R-module J is injective if every morphism f from the domain of a monomorphism can be extended to the codomain; that is, if for each monomorphism $m: A \to B$, any $f : A \to J$ extends to an $f': B \to J$ with $f'm = f$, as in the commutative diagram



In this case the existence of injectives is not so evident, except in the case of abelian groups (Z-modules) where the injectives are exactly the divisible abelian groups. However in this case it was known that every abelian group could be embedded in an injective (i.e., divisible) abelian group. In 1940 R. Baer, using transfinite induction, proved that the same held for R-modules over every ring. This was exactly the result necessary to construct an injective resolution for any R-module.

In 1953, Eckmann and Schopf provided a new and much more perspicuous proof that every R-module A could be embedded in an injective one. They first embedded A, regarded as an abelian group, into a divisible group D and then formed the double embedding

$$A \rightarrowtail \hom(R, A) \rightarrowtail \hom(R, D)$$

proving that D divisible meant that the hom (R, D) is injective. Going beyond this, they observed that there was in fact a *minimal* way of embedding A into an injective module J. Finding this depended on the notion of an essential extension. A submodule $A \subset B$ or a monomorphism $A \rightarrowtail B$ is *essential* if for each submodule S of B, $S \cap A = 0$ implies $S = 0$; in other words $B \supset A$ is essential if every non-trivial submodule of B must actually meet A in some non-zero elements. From this definition it is not hard to see that each module A has a *maximal* essential extension $A \rightarrowtail E$. This maximal essential extension now turns out to be the minimal injective extension of A—a result of great beauty and use.

13. Functors and Categories

In another direction, the development of the cohomology of groups was an essential preliminary to the formulation of the notions of category and functor. Hopf's discovery of the second homotopy group $H_2(G, \mathbf{Z})$ provided a highly non-trivial example of a functor of G. To be sure, this functor had been present before; in the form

$$H_2(G, \mathbf{Z}) = R \cap [F, F] / [F, F] \qquad G = F/R\,,$$

it was in fact identical with Schur's "multiplicator"—though any general description of "functors" would have been unlikely at the time when Schur was using his multiplicator in connection with projective representations. However, in 1942 the mathematical atmosphere was different and more ready for abstractions (thanks to the influence of Hilbert, Emmy Noether, and others). Moreover, there were other prominent examples of non-trivial constructions on groups which were functors—the group Ext (G, A) of all abelian extensions of the abelian group A by G being one. Indeed, it was principally this functor (as it was needed for the universal coefficient theorem in cohomology) that led Eilenberg-Mac Lane in 1943 to the step of introducing categories in general and functors on them, both covariant and contravariant.

The categorical language was soon generally used for homology theory and homological algebra—but one essential element of that language was missing: The notion of adjoint functor. This notion did not actually appear till D. M. Kan's clear introduction in 1958. To be sure, many special examples, usually under the form of a suitable universal property, had been long present. However, the great merit of the notion lies in its generality and systematic availability. In retrospect (see Mac Lane [1976]) it is strange indeed that it took 15 years from the introduction of categories in 1943 to the definition of adjoint functors in 1958. It may indeed be that there was a widespread prejudice against very general notions ("general abstract nonsense") and that the mores of mathematical research were determined more by a sort of positivistic view—all that matters are hard calculations leading to explicit theorems solving known problems. This clearly useful and effective standard—for most mathematical purposes—may have needlessly inhibited the development of appropriate general concepts. This is hard to judge with certainty. I do know that Eilenberg-Mac Lane for a dozen years after their initial publication on category theory considered that category theory was chiefly a language, and that further serious research in the subject was not worth trying. When Daniel Kan, coming from outside the main communities of mathematics, did arrive at the notion of a pair of adjoint functors, his work was warmly greeted by Eilenberg.

This may leave us to wonder if there are other general notions not yet discovered which might be useful for the organization of mathematics.

14. Duality

One general notion, that of categorical duality and its topological application, did not lack for attention. Pontryagin duality for topological groups had long (since about 1930) been a central tool for the algebraic topologists, especially for its use with the coefficient groups of knowledge and cohomology. The alternative possibility of dualities which are axiomatic (because they arise from a dual involution of the undefined terms of an axiom system) could not very well become relevant for topology until the categorical language was available. Possibly the first step in this direction was the proof (about 1940) by Reinhold Baer that the dual of a free group (in effect, the dual taken in the category of all groups) was necessarily a one-element group. This result may even have had some political overtones, since the dual of "free" might then have been labelled "fascist".

In 1948 Mac Lane, during a four-month stay in Zurich, observed that the use of categories would allow the exact formulation of the notion of the dual of a theorem about a category—by reversing both the arrows and the composition in the statement (in presently more fashionable terminology, by taking the original theorem for the opposite category). Mac Lane's first paper on this subject, in the *Proceedings* of the National Academy of Sciences, dealt chiefly with such dualities for the category of groups. This study did not lead very far, because the duals of many true theorems in this category are not true—and one has till this day no real understanding of the class of theorems on groups for which such duality would hold. Mac Lane's second paper [1950] on this topic was concerned more with categorical ideas, especially the introduction of what is essentially the notion of an abelian category (his axioms were too clumsy because he tried to get an exact duality between subobjects and quotient objects; later it became clear that duality "up to isomorphism" suffices). This *should* have even been clear at the time; specifically, the same paper presented the (now familiar) categorical definition of direct product and free product—a definition by diagrams which identifies these products only "up to isomorphisms".

Duality considerations for the category of topological spaces turned out to be much more profitable. The essential observation here is that the covering homotopy theorem (and consequently, the notion of a fiber map) is the dual of the homotopy extension theorem (and the notion of a cofiber map). I have not succeeded in determining who first observed this duality, but it is clear that the team of Eckmann and Hilton most effectively formulated this idea (in their terms, projective and injective homotopy). This they began with three notes in the *Comptes Rendus* in 1958, and continued in a considerable sequence of papers, in particular, the three papers [1962-1963] on group-like structures in general categories. Of these, the first 1958 note considered group structure on the set $\Pi\,(A, B)$ of homotopy classes of maps of the space A into the space B. They proved that an H-space structure on B gave a group structure on $\Pi\,(A, B)$ which is natural in A and dually that a H'-space structure on A yields a group structure on $\Pi\,(A, B)$ which is natural in B. Here too they proved the beautiful easy theorem that for A an H'-space and B an H-space the two group structures on $\Pi\,(A, B)$ agree and are abelian, observing the consequence that higher homotopy groups are abelian. They used systematically the reduced suspension Σ, the loop space construction Ω and the adjunction

$$\Pi(\Sigma A, B) \cong \Pi(A, \Omega B)$$

(though they did not explicitly note that this made Σ left adjoint to Ω). They went on to define higher homotopy groups

$$\Pi_n(A, B) = \Pi(\Sigma^n A, B) = \Pi(A, \Omega^n B)$$

corresponding relative groups and the appropriate long exact sequences. These long exact sequences, which extended Barratt's 1955 "track group sequences" were further codified by D. Puppe and are now the Puppe sequences. Eckmann's report at the 1962 International Congress gives an especially clear formulation of this whole set of ideas (including the notion of spectra).

Our main contention is that the systematic use of cohomology of groups and the resulting categorical ideas inevitably led to the systematic use of duality in algebraic topology. We have not tried here to trace the exact authorship of these ideas—because it is clear that many topologists played a role in this work. John Moore was concerned with Eilenberg-Mac Lane spaces $K(\Pi, n)$—the spaces arising from the cohomology of groups with only one homotopy group Π in dimension n; in the 1954 Cartan seminar he introduced the (quasi-dual) Moore spaces $K'(\Pi, n)$—with only one homology group Π in dimension n. At about that time he and others must have considered the "duals" of the Postnikov decomposition of a map—a notion explicitly formulated in the fourth Eckmann-Hilton note in *Comptes Rendus* (1959). E. H. Brown's work (1962) on the Representation of Cohomology Theories, and George Whitehead on Generalized homology theories (1962), also belong here. These ideas were surely "in the air".

One historical note on these ideas did turn up during the Zurich conference. Given a cohomology theory h^* defined by a spectrum B and given a polyhedron A, there is a spectral sequence E_n^{pq} starting with the ordinary cohomology $E_2^{pq} = H^p(A, h^q(S_0))$ and converging to (the graded module associated to a filtration of) $h^{p+q}(A)$. This spectral sequence is usually called the Atiyah-Hirzebruch spectral sequence, because it first appeared in print for the case when h^* is K-theory in a paper (1961) by these authors. The background, as told me by J. F. Adams, is as follows: On August 4, 1955, George Whitehead has submitted to the *Transactions* of the American Mathematics Society a paper (1956) on the homotopy groups of joins and unions. In modern language, it gave for stable homotopy Π_*^S a spectral sequence $H_* (X, \Pi_*^S Y) \Rightarrow \Pi_*^S (X * Y)$, where $X * Y$ is the join of the spaces X and Y. In discussion with Adams, Whitehead talks about his definition of a generalized homology theory K and said that his paper "should" have proved $H_* (X, K_* (\text{pt})) \Rightarrow K_* (X)$. Later, Atiyah told Adams about his joint work with Hirzebruch on K-theory as a generalized cohomology; he also wondered about its relation to ordinary cohomology. Adams, recalling the words of Whitehead, observed that there was a suitable spectral sequence; Atiyah asked how it was constructed and whether it was published. Adams thus reported that it was constructed in the inevitable way, from an appropriate filtration—but that it had not been published. Atiyah resigned himself to the trouble of writing it up—and so it is now called the Atiyah-Hirzebruch sequence. Given the familiarity at that time with the technique of spectral sequences, it is clear that this sequence was sure to be discovered at about that time—if not by one author, then by another.

15. Cohomology of Algebraic Systems

The cohomology of groups was just the starting point for the study of corresponding cohomology theorems of other sorts of algebraic systems. A few months after the discovery of the cohomology of groups, Hochschild found a corresponding cohomology for algebras. Again, the 2-dimensional cohomology group of an algebra corresponded to an extension problem for algebras, and it soon turned out that the Eilenberg-Mac Lane interpretation of H^3 as obstructions for non-abelian extensions of groups could also be carried over to algebras. Presently Chevalley and Eilenberg formulated a cohomology theory for Lie algebras. It was now amply clear that the idea of cohomology, originally conceived as a measure of the connectivity of spaces, was also relevant as a record of some of the aspects of quite a variety of algebraic systems. The connection with topology remained strong, however. For example, the Eilenberg-Mac Lane spaces $K(\Pi, n)$ were defined topologically, as spaces with Π the only non-vanishing homotopy group- in dimension n; their stable cohomology, however, could be interpreted as the cohomology of the abelian group Π (Mac Lane [1950]). This cohomology—and that of other algebraic systems—can be calculated systematically from a complex which is "generically acyclic" in the sense of Eilenberg-Mac Lane [1951] [1955]. The full meaning of this notion is still mysterious.

Subsequently these cohomology theories were unified and organized in a striking fashion by the notion of triple cohomology. This idea was an outgrowth of the notion of a pair of adjoint functors $F : X \to A$ and $U : A \to X$. Eilenberg and Moore observed that the composite endofunctor $T = UF : X \to X$ inherited from the given adjunction not only the "universal" natural transformation $\eta : I \to T$ but also a natural transformation $\mu : T^2 \to T$, with formal properties parallel to those of the multiplication μ and the unit η of a monoid or of a ring. The quadruple $< X, T, \eta, \mu >$ with these properties they called a triple, and they constructed the category of "algebras" for such a triple (better monad), to match exactly the actions of a monoid or the modules over a ring. Soon afterwards, Barr and Beck observed [1966] [1969] that these monads and these algebras could be used to systematically construct the cohomology of groups, modules, algebras and other algebraic systems. The resulting "triple cohomology" or "cotriple cohomology" was beautifully developed in an extensive seminar at the Forschungs Institut of the E.T.H. at Zurich. This development (recorded in part in a Springer Lecture Notes Vol. 80) in particular finally accounted systematically for the central role of the bar construction in all these cohomologies—thus bringing to full understanding exactly the construction first used by Eilenberg-Mac Lane to introduce the cohomology of groups. Eckmann's timely encouragement of this triple cohomology development at Zurich is another one of his major contributions to mathematics.

16. Some Historical Questions.

Our discussion has traced some of the ramifications of the development of the cohomology of groups. Inevitably it raises for consideration a number of speculative questions—which can hardly be settled by reference to this one sample piece of the history of recent mathematics.

First, a mathematical idea looks very different coming and going. The cohomology of groups started as a particular question as to a construction of part of the 2-dimensional homology groups. It also may have started as a construction to realize explicitly the meaning of that theorem of Hurewicz asserting that the fundamental group of an aspherical space determines all the homology groups. Thus the cohomology of groups, intended to provide the solution to a problem, became a theory and also became a connection (or, the discovery of a connection) between algebra and topology. This discovery came (by chance or by direct influence) at a time which was ripe for such discoveries, because of the movements to make algebra abstract and to algebricize topology.

What started as a problem became a theory and this led to problems again: What are the groups of cohomological dimension one? (By Stallings and Swan, just the free groups). What is the full algebraic interpretation of $H^4(G, A)$ (still a mystery)? Is Whitehead's conjecture true? If $\mathrm{Ext}(G, \mathbf{Z}) = 0$ for given G and the abelian group $\mathbf{Z}$, is G free? (answer, by Shelah, maybe yes or maybe no, depending on your set theory (see Ecklof [1976])). There appears to be a movement in mathematics from problem to ideas to theories to problems to counterexamples—and back again.

Are there breakthroughs of complete novelty? Not quite. As we have argued, there are decisive papers, like the 1942 paper of Hopf which started our whole subject. There were striking new ideas in that paper, but they were not unprecedented; rather, they were rooted, as we have noted, in earlier studies on homotopy and on the homology of Lie groups. Hopf's paper was a new idea, but one built on an older one, hence not a new paradigm. With such a new idea, other developments, here the higher dimensional cohomology, became inevitable—as their multiple discovery shows. In this case, the development came soon; that is not always so, as witness the long wait before the "inevitable" development of the notions of adjoint functors. With the inevitable developments, there are also some which are evitable: They were not needed and they don't seem to matter. It is well known that there are many such papers; just by way of a constructive existence proof, I cite the 1947 paper by Eilenberg and Mac Lane in which the higher cohomology groups $H^n(G, A)$ were interpreted by non-associative multiplications. This result seems to have found no use; no matter, the exploration of the unknown is sure to lead us up some false paths.

Finally, our small piece of history shows that the development of mathematics is by no means single-minded. It involves the interaction between the ideas of many individuals and the interpenetration of different fields. In the present case, the interplay between algebra and topology is prominent, and is typical of the contributions of Beno Eckmann to our science.

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