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ALGEBRAIC ASPECTS OF THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS [1]

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This is intended to be a report on some recent work in the theory of linear partial differential equations with analytic coefficients. The point of view is here to focus attention, not mainly on the solutions of the equations, but on the structure of the system of equations itself and more precisely on the module over the ring of differential operators defined by this system. The result is a kind of non-commutative algebraic (or better, analytic) geometry, which is rapidly growing now; as one will see from the references, the main contributor is M. Kashiwara.

In this report, we will limit ourselves to the **C**-analytic case, and therefore omit the applications to analysis, which require, of course, looking at the **R**-analytic case. We will mention also very briefly the fundamental tool of "microlocalization", or localization in the cotangent space; but the reader should not forget that this localization plays a fundamental role in the theory, and in many proofs (f.i. in the proof of the "involutiveness of characteristics"), and should therefore be much more fully developed in a systematic exposition.

1. DIMENSION OF $\mathscr{D}$-MODULES

Let X be a **C**-analytic manifold, and n its dimension. We denote by $\mathcal{O}_X$ (or $\mathcal{O}$) the sheaf of holomorphic functions on X, and by $\mathscr{D}_X$ (or $\mathscr{D}$) the sheaf of linear differential operators on X with coefficients in $\mathcal{O}$; we denote by $\mathscr{D}_k$ the subsheaf of operators of degree $\leqslant k$, and by $\mathscr{D}'_m$ the subsheaf of $\mathscr{D}_m$ of operators without constant term; as it is well-known, $\mathscr{D}_0$ can be identified with $\mathcal{O}$, and $\mathscr{D}'_1$ with the sheaf of vector fields on X; moreover, if we denote by $T^*X \xrightarrow{\pi} X$ the cotangent bundle of X, then $\operatorname{gr} \mathscr{D} = \oplus \mathscr{D}_m / \mathscr{D}_{m-1}$ is naturally isomorphic to the subsheaf of $\pi_* (\mathcal{O}_{T^*X})$ of functions "polynomial with respect to the variables of the fibre". More explicitly, if U is an open set of X admitting local coordinates $x_1, \ldots, x_n$, then

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$\Gamma(U, \operatorname{gr} \mathscr{D}) = \Gamma(U, \mathcal{O})[\xi_1, ..., \xi_n]$, where ξ_i is the image of $\partial_i = \dfrac{\partial}{\partial x_i}$ in gr $\mathscr{D}$. From this results easily that $\mathscr{D}$ as sheaf of rings is left and right coherent.

Let M be a "system of p.d.e." on X, i.e. a coherent left $\mathscr{D}$-Module; a filtration on M is an increasing sequence of sub $\mathcal{O}$-Modules M_k verifying $M = \cup M_k$, $D_l M_k \subset M_{k+l}$ for all l, k; the filtration is called "good" if the two following conditions are satisfied

(GF 1) For every k, M_k is coherent over $\mathcal{O}$.

(GF 2) There exists $k_0 \in \mathbf{N}$ such that $D_l M_{k_0} = M_{k_0+l}$, for every $l \in \mathbf{N}$.

Locally, any coherent left $\mathscr{D}$-Module M admits a good filtration; now, one defines the "characteristic variety" of M, char M as follows: choose locally a good filtration $\{ M_k \}$, and consider gr M; as gr $\mathscr{D}$-Module, it is coherent, and therefore its support V in T^*X is well-defined, as an analytic subset of T^*X, relatively algebraic and homogeneous with respect to π, i.e. with respect to variables of the fiber. Note that gr M could depend on the (good) filtration we have chosen; but it turns out that V is independent of it, as is the multiplicity of gr M at any point of V. By definition, we have $V = \operatorname{char} M$, and $\dim_a M = \dim_a V$ ($a \in T^*X$).

In what follows, we identify X with the 0-section of T^*X; due to the homogeneity of V, if $\pi(a) = x$, one has: $\dim_x M \geqslant \dim_a M$. The first nontrivial result of the theory is the following

THEOREM 1.1 (Bernstein [1] — Björk [2] — Kashiwara [8]). *At any point* $x \in X \cap V$, *one has* $\dim_x M \geqslant n$.

A simple proof is given in [B.L.M.] (it is probably the same as Kashiwara's).

A much deeper result, which was conjectured by Guillemin-Quillen-Sternberg and proved in some special cases by these authors, is due to Sato-Kawai-Kashiwara [S.K.K.]; denote by λ the Liouville form on T^*X (in local coordinates, $\lambda = \Sigma \xi_i dx_i$), and put $\omega = d\lambda$; then ω defines canonically a symplectic structure on T^*X; this structure is related to p.d.e. in the following way: Let P and Q be differential operators of orders p and q respectively, and let $\sigma(P)$, $\sigma(Q)$ be the "symbols" of P and Q, i.e. the images of P and Q in gr $\mathscr{D}$; then $[P, Q] = PQ - QP$ is an operator of order $p + q - 1$ and one has $\sigma[P, Q] = \{ \sigma(P), \sigma(Q) \}$, the Poisson bracket of $\sigma(P)$ and $\sigma(Q)$ with respect to ω. Recall also that an analytic subset V of a symplectic manifold is called "involutive" if the sheaf of functions vanishing on V is stable under Poisson bracket. Then, one has

THEOREM 1.2. *The characteristic variety of any coherent $\mathscr{D}$-Module is involutive.*

The proof given in [S.K.K.] is difficult, and uses "pseudo differential operators of infinite order". Recently, a simpler proof, using only usual pseudo differential operators has been obtained independently by Kashiwara and the author (see a forthcoming lecture in "Séminaire Bourbaki").

As consequences of th. 2, and standard facts of symplectic geometry, one has, for every $a \in V = \text{char}\, M : \dim_a M \geqslant n$; moreover, if $\dim_a V = n$, then V is *lagrangian* near a, i.e., on the smooth part V_s of V, in the neighbourhood of a, one has $\lambda / V_s = 0$. If V is globally of dimension n, and therefore globally lagrangian, then there exists a unique stratification of $U = X \cap V$ into smooth submanifolds U_α such that $V = \bigcup_\alpha \overline{N^* U_\alpha}$, $N^* U_\alpha$ the conormal bundle of U_α in X.

Definition 1.3. A coherent left $\mathscr{D}$-Module M is called "holonomic" (or "maximally overdetermined") *if* $\dim M = n$ *(or, equivalently, if* char M *is lagrangian*).

Using the properties of the multiplicity, one sees the following: if $\dim_x M = n$, i.e. if M is holonomic at x, then M_x is a $\mathscr{D}_x$-module of finite length. Therefore, the holonomic M play in this theory more or less the same role as the closed points and the modules of finite length in algebraic geometry. But their structure is much less known! For instance, in the case $n = 1$, analyzing locally that structure is (essentially) equivalent to classifying differential equations near a singularity, regular or irregular. We mention here that recent progress have been made in that problem; we do not insist on that, which is beyond the scope of this report.

To end this section, a few words on pseudo-differential (or "micro-differential") operators. In the C^∞-case, they are well-known; in the analytic case, they were defined by Boutet de Monvel-Krée [3], and studied systematically in [S.K.K.] in connection with hyperfunctions and microfunctions (the reader who is only interested in microdifferential operators could perhaps read independently chap. II of [S.K.K.], and also a partial exposition in [B.L.M.]). They are defined roughly as follows; let U be an open set in T^*X, and choose local coordinates $x = (x_1, \ldots, x_n)$, $\xi = (\xi, \ldots, \xi_n)$ in U. We define $\mathscr{E}(U) = \{ p_j(x, \xi) \}_{j \in \mathbf{Z}}$, $p_j \in \mathscr{O}_{T^*X}(U)$, such that

M.1) p_j is homogeneous of degree j in ξ.

M.2) $\sup |p_j(x, \xi)| \leqslant (-j)!\ R_K^{-j}$ for any $K \subset\subset U$, and $j < 0$.

M.3) $p_j = 0$ for $j >> 0$,

At a point $(x, 0)$, the p_j are homogeneous polynomials of degree j in ξ; therefore, $p_j = 0$ for $j < 0$, and, for $j \geqslant 0$, p_j can be identified with the differential operator $p_j(x_i, \partial_i)$; the formulae for multiplication and change of variables in $\mathscr{E}$ are chosen in order to extend what happens on $\mathscr{D}$. In that way, one get a sheaf $\mathscr{E}$ on T^*X with a filtration $\mathscr{E}_j, j \in \mathbf{Z}$ and a structure of (flat) $\pi^*(\mathscr{D})$-Module. All the properties of $\mathscr{D}$ mentioned before can be extended to $\mathscr{E}$, which is called the sheaf of (convergent) microdifferential operators. Note also the following property: if $p \in \mathscr{E}(U)$ has a symbol $\sigma(p)$ which does not vanish, then p is invertible in $\mathscr{E}(U)$ [3]; from that results easily the following useful property: if M is a coherent $\mathscr{D}$-Module, one has char M = support of $\tilde{M}$ with $\tilde{M} = \mathscr{E} \otimes_{\pi^{-1}\mathscr{D}} \pi^{-1} M$.

A variant of the preceding sheaf with essentially similar properties, is given by the sheaf $\hat{\mathscr{E}}$ of "formal" microdifferential operators (it is defined like $\mathscr{E}$, by just removing M.2). Perhaps this sheaf, or an algebraic counterpart, could have some interest for an *algebraic* theory of $\mathscr{D}$-Modules.

2. General constructions on $\mathscr{D}$ and $\mathscr{E}$-Modules

(2.1) *Canonical transformations.*

This operation is restricted to $\mathscr{E}$-Modules on open sets $U \subset T^*X - X$; this is the analytic counterpart of Maslov's ideas [13] and of the theory of "Fourier integral operators" by Hörmander [7]. Given a homogeneous symplectic diffeomorphism $U \xrightarrow{\varphi} V$, with $U, V \subset T^*X - X$, there exists a (non-unique) isomorphism $\mathscr{E}\,|\,U \to \mathscr{E}\,|\,V$, which respects the filtrations, and verifies $\sigma\Phi(P) = \sigma(P) \circ \varphi^{-1}$. This is often useful to reduce the support of an $\mathscr{E}$-Module, at least at smooth points, to canonical form. Although this is a very fundamental ingredient of the theory, we will not insist on it here. We just mention that Φ is defined by a suitable holonomic system, whose support (= characteristic variety) is precisely the graph of φ in $U \times V$. For the details, we refer to [S.K.K.]; see also [B.L.M.].

(2.2) *Direct images.*

We introduce first some definitions; let Y be another manifold of dimension p, and $f: X \to Y$ a holomorphic mapping; we define $\mathscr{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathscr{D}_Y)$; this sheaf on X is nothing but the sheaf of differential operators from $f^{-1}(\mathcal{O}_Y)$ into $\mathcal{O}_X$; therefore it has a structure of left $\mathscr{D}_X$-Module and right $f^{-1}(\mathscr{D}_Y)$-Module; we leave to the reader the explicit definition of these structures. Similarly note that Ω_X^n, the sheaf of holomorphic n forms on X is a right $\mathscr{D}$-Module (by the following action, if ξ is a vector field, and $\alpha \in \Omega^n$, we write $\alpha\xi = -\theta_\xi \alpha$, θ the Lie derivative); we define therefore the sheaf on X of $(f^{-1}(\Omega_Y^p), \Omega_X^n)$-differential operators by $\mathscr{D}_{Y\leftarrow X} = f^{-1}(\mathscr{D}_Y \otimes_{\mathcal{O}_Y} (\Omega_Y^p)^{-1}) \otimes_{f^{-1}(\mathcal{O}_Y)} \Omega_X^n$ [here we use the right structure of $\mathscr{D}_Y$-Module over $\mathcal{O}_Y = \mathscr{D}_{0Y}$]; it has a structure of right $\mathscr{D}_X$-Module and left $f^{-1}(\mathscr{D}_Y)$-Module.

Now, let M be a left coherent $\mathscr{D}_X$-Module; the direct images (or "integration" in the fiber) are defined by $\int^i M = R^i f_* (\mathscr{D}_{Y\to X} \otimes^L_{\mathscr{D}_X} M)$, where R (resp. L) denotes the right (resp. left) derived functors. To understand the meaning of these operations, we will examine special cases.

i) *Case where Y is a point* (the "absolute" case).

Here, one has $\mathscr{D}_{Y\leftarrow X} = \Omega_X^n$; on the other hand, denote by $DR^\cdot(M)$ the "de Rham complex of M"

$0 \to M \to M \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{d} \dots \xrightarrow{d} M \otimes_{\mathcal{O}_X} \Omega_X^n \to 0$, where d is the usual exterior derivative; it is easy to verify that one has an isomorphism $\Omega_X^n \otimes^L_{\mathscr{D}_X} M \xrightarrow{\sim} DR^\cdot(M)[n]$ (where $[n]$ means "shifted n times to the left"); and, one has also an isomorphism $DR^\cdot(M) \simeq \underline{\underline{R}}\,\mathrm{Hom}_{\mathscr{D}_X}(\mathcal{O}_X, M)$; therefore, one has

$$\int^i M = \underline{\underline{H}}^{i+n}(X, DR^\cdot(M)) = \mathrm{Ext}^{i+n}_{\mathscr{D}_X}(X; \mathcal{O}_X, M).$$

Therefore, here, the direct image is the global de Rham hypercohomology of M, with a shifting of the degree by n.

ii) *The case where $X \to Y$ is smooth* (i.e. is locally a submersion).

This case is similar: one gets the relative de Rham cohomology (with a shifting by $n-p$). Note that we have automatically a structure of left $\mathscr{D}_Y$-Module on $\int^i M$; in the case where $M = \mathcal{O}_X$, this structure is just the so-called "Gauss-Manin connection".

iii) *The case where X is a submanifold of Y.*

In local coordinates, we can suppose $p = n + k$, $x_i = y_i$, $1 \leqslant i \leqslant n$, and that X is defined by $y_i = 0$, $i \geqslant n + 1$. Then one has

$\mathscr{D}_{Y \leftarrow X} \simeq \mathscr{D}_Y / \sum_{i \geq n} \mathscr{D}_Y y_i$; elements of $\mathscr{D}_{Y \leftarrow X}$ can be written as $\Sigma\, \mathscr{D}_{y'}^{\alpha} . P_\alpha$, with $y' = (y_{n+1}, ..., y_p)$, $P_\alpha \in \mathscr{D}_X$; this is a free Module over $\mathscr{D}_X$, and f_* here is exact; therefore $\int^i M = 0$, $i \neq 0$, and $\int^0 M$ is just the $\mathscr{D}_Y$-Module whose elements can be written uniquely as $\Sigma\, D_{y'}^{\alpha} \otimes m_{\alpha'}$ $m_\alpha \in \mathscr{D}_X$. This is just the same correspondence as in the theory of distributions: "distributions on X" $\to$ "distributions on Y with support in X".

Now, one can prove that formation of $\int^{\cdot}$ is compatible with composition (i.e. one has an isomorphism of derived functors $\int_{fg}^{\cdot} \simeq \int_f^{\cdot} \cdot \int_g^{\cdot}$); then, the general case reduces to ii) and iii).

The following theorem is due to Kashiwara [10].

THEOREM 2.2.1. *Suppose f projective* (i.e. *proper and factorizing through some closed embedding $X \to Y \times \mathbf{P}_k(\mathbf{C})$, and suppose that M has a global good filtration. Then*

i) *The $\int^i M$ are coherent $\mathscr{D}_Y$-Modules.*

ii) *The characteristic variety of $\int^i M$ is contained in the set of $\eta \in T^* Y$, with $y = \pi(\eta)$, such that there exists $\xi \in$ char M, with $x = \pi(\xi) \in X$ verifying $y = f(x)$, $\xi = Tf_x^*(\eta)$; here Tf^* denotes the cotangent map of f.*

If M is holonomic, and the other hypotheses of the theorem are satisfied, this implies easily that the $\int^i M$ are holonomic.

The proof of ii) requires some microlocalization of the notion of direct images, which I will not develop here. Also, it is likely that the hypothesis "f proper" is sufficient for the conclusions of the theorem. Perhaps, it is also true that one has coherence of local direct images of *holonomic* Modules, when one replaces X by a small ball, as in Milnor's work on singularity of hypersurfaces, and in the study of local Gauss-Manin connection by several authors (Brieskorn [4], Hamm [6], etc.); this is at least true in the absolute case (see § 3).

(2.3) *Inverse images, and localization.*

Let $f: X \to Y$, as before, and M a coherent left $\mathscr{D}_Y$-Module; as in analytic geometry, one defines $f^* M = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(M)$; the obvious isomorphism:

$f^* M \simeq \mathcal{D}_{X \to Y} \otimes_{f^{-1}(\mathcal{D}_Y)} f^{-1}(M)$ provides $f^* M$ with a structure of $\mathcal{D}_X$-Module; the left derived functors $L_i f^* M$ are defined in the same way, with "Tor".

Again, the study is reduced to two cases: i) submersions, ii) closed embeddings; the first case is trivial, therefore we consider only the second and suppose, from now on, that X is a closed submanifold of Y. In that case, the $L_i f^* M$ are not coherent in general (take for instance $M = \mathcal{D}_Y$, and X defined by $y_p = 0$). There are three cases of interest:

a) *The non-characteristic case.*

One says that X is non-characteristic with respect to M if char $M \cap N^*X$ is contained in the zero-section (N^* denotes the conormal bundle). This is a well-known notion, f.i. in connection with the Cauchy-Kovalevskaya theorem. Then, if X is non-characteristic, $f^* M$ is coherent, and $L_i f^* M = 0$, $i \geqslant 1$. Moreover, one has char $f^* M = (Tf)^*$ (char M). See [S.K.K.].

b) *The case where* M *has support in* X.

In that case, one has $L_i f^* M = 0$, $i \neq d = p - n$ and $L_d f^* M$ is a coherent $\mathcal{D}_X$-Module; we will denote it by $\bar{f}^* M$; in local coordinates, $x_i = y_i$, $1 \leqslant i \leqslant n$, X defined by $y_{n+1} = \ldots = y_p = 0$, $\bar{f}^* M$ is the set $\overline{M}$ of $m \in M$ annihilated by $y_{n+1}, \ldots, y_p$ (take the resolution of $\mathcal{O}_X$ over $\mathcal{O}_Y$ by the Koszul complex), but this is not intrinsic; $\overline{M}$ has no canonical structure of $\mathcal{D}_X$-Module, and has to be tensorised by a suitable invertible sheaf on $\mathcal{O}_X$ to become $\bar{f}^* M$.

One remarkable phenomenon occurs: M is canonically isomorphic with $\int^0 \bar{f}^* M$; in other words, the functors $M \mapsto \bar{f}^* M$ and $N \mapsto \int^0 N$ give an equivalence between the category of coherent $\mathcal{D}_Y$-Modules with support in X and the category of $\mathcal{D}_X$-Modules, a situation much simpler than in usual analytic geometry. For instance, in local coordinates, the coherent $\mathcal{D}_X$-Modules with support o are finite sums of copies of $\mathcal{D}_X \delta \simeq \mathcal{D}_X / \Sigma \mathcal{D}_X x_i$ (this module is also well-known to algebraists as the injective envelope of $\mathbf{C}$ over $\mathbf{C}\{x_1, \ldots, x_n\}$).

One has $\dim N - n = \dim \int^0 N - p$; in particular, holonomy is preserved in this correspondence. For these results, see [8] or [B.L.M.].

c) *The case where* M *is holonomic.*

In this case, one has the following result, much more difficult than the preceding ones:

THEOREM (2.3.1). *If* M *is holonomic on* Y, *then the* $L_i f^* M$ *are holonomic on* X (Kashiwara [11]).

However, one problem here is to find the characteristic varieties (this restriction seems to have no microlocal counterpart). Note also that, in the case of modules over the Weyl algebra, i.e. the algebra of differential operators on $\mathbf{C}^n$ with *polynomial* coefficients, the holonomy of $f^* M$ was proved previously by I.N. Bernstein [1].

The preceding theorem can be stated in a more general context, using local cohomology. If now Z is a closed analytic subset of Y, defined by a coherent $\mathcal{O}_Y$-Ideal J, we define $H^i_{[Z]} M = \varinjlim \operatorname{Ext}^i_{\mathcal{O}_X} (\mathcal{O}_X/J^k, M)$; this is not the "transcendental" local cohomology $H^i_Z M$, but the analytical translation of the local cohomology of schemes; it is easily provided with a structure of $\mathcal{D}_Y$-Module. Now, if $X \subset Y$ is a submanifold, it is easy to prove that one has $L_i f^* M = \bar{f}^* (\underline{H}^{d-i}_{[X]} M)$, with $d = \operatorname{codim}_Y X$. Therefore, theorem (2.3.1) is a special case of the following theorem (same reference):

THEOREM (2.3.3). *If* M *is holonomic, then the* $H^i_{[Z]} M$ *are holonomic.*

As an easy consequence, the sheaf of meromorphic sections of a connection with singularities in the sense of Deligne [5] is a holonomic $\mathcal{D}$-Module. In some sense, they are the "general case" of holonomic $\mathcal{D}$-Modules (a problem is to give a meaning to this assertion). In particular, modulo non-singular compactifications, one deduces immediately from that fact the following theorem, proved previously by Björk (unpublished ?): the algebraic de Rham cohomology of an algebraic connection on an affine non-singular $\mathbf{C}$-variety is finite.

3. FURTHER RESULTS ON HOLONOMIC SYSTEMS

First, note that, if M is a coherent left $\mathcal{D}$-Module on X and N any $\mathcal{D}$-Module, then $\operatorname{Hom}_{\mathcal{D}} (M, N)$ can be interpreted as the set of solutions of the system of p.d.e. defined by M, with values in N (for instance, if J is a left coherent sheaf of ideals of $\mathcal{D}$, and $M = \mathcal{D}/J$, then $\operatorname{Hom}_{\mathcal{D}} (M, N)$ is the set of $n \in N$ annihilated by J). For instance, taking $N = \mathcal{O}$, we get the holomorphic solutions of M; on the other hand, we have seen the relation between $\mathcal{R} \operatorname{Hom} (\mathcal{O}, N)$ and the de Rham cohomology of N.

Note also that $\mathcal{O}$ is holonomic as a $\mathcal{D}$-Module (one has $\mathcal{O} = \mathcal{D}1 \simeq \mathcal{D}/\Sigma\mathcal{D} \dfrac{\partial}{\partial x_i}$ and therefore char $(\mathcal{O}) = X$, the null section). This explains the interest of the following theorem, due again to Kashiwara [9], [11].

THEOREM 3.1. *If M and N are holonomic, then the sheaves* $\mathrm{Ext}^i_{\mathcal{D}}(M, N)$ *are* **C***-analytically constructible* (i.e. *there exists a* **C***-analytic stratification of X, such that, on each stratum, the sheaf is locally isomorphic to the constant sheaf* $\mathbf{C}^l$ *for some* l); *in particular, the fibers* $\mathrm{Ext}^i_{\mathcal{D}}(M, N)_x$ *are finite over* **C**.

Another problem is posed by the systematisation and extension to $\mathcal{D}$-Modules of the known theorems on regular connexions [5]. Here, one needs some regularity assumptions (for instance, the algebraic cohomology of a connection on an affine non-singular algebraic variety is the same as the analytic one when the connection is regular at infinity, but not in general). This subject is rapidly developing at the moment, and we will only mention some references:

a) In Kashiwara-Oshima [12], one will find regular $\mathcal{D}$- or $\mathcal{E}$-Modules, defined, and studied at generic points of the characteristic variety.

b) In Mebkhout [14] and Ramis [15], one will find systematic developments of the Grothendieck comparison theorem, in relation with $\mathcal{D}$-Modules and also with the "Cousin complex" of Grothendieck, in the analytical version of Ramis-Ruget [16].

Finally, I mention that, recently, Kashiwara and Kawai have announced an extension of the comparison theorem to any regular holonomic $\mathcal{D}$-Module.

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