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#### 4. OUTLINE OF THE PROOF OF THEOREM 5

Fix  $a \in \left(0, \frac{1}{8\pi}\right)$  and let  $D = \overline{\mathbb{C}} - \gamma$ , where

$$\gamma = \{z = \pm i e^{(-a+i)t} : 0 \leq t < \infty\} \cup \{0\}.$$

Then  $D$  is a simply connected domain which contains the disjoint  $e^{2\pi a}$ -spirals

$$\alpha = \{z = e^{(-a+i)t} : 0 < t < \infty\}, \quad \beta = \{z : -z \in \alpha\}.$$

Next let  $f$  denote any conformal mapping of  $D$  which fixes the points  $1, -1, \infty$ . To complete the proof of Theorem 5 it is sufficient to show that there exists a positive constant  $\delta = \delta(a)$  such that  $f(D)$  is not a Jordan domain whenever  $\|S_f\|_D \leq \delta$ . This is done in three steps.

First using Lemma 1 and a normal family type argument, we can prove that there exists a  $\delta_1 = \delta_1(a) > 0$  with the following property. If  $\|S_f\|_D \leq \delta_1$ , then  $f(\alpha)$  and  $f(\beta)$  are  $b$ -spirals from 1 onto  $z_2$  and from  $-1$  onto  $w_2$ , respectively, where  $b \in (1, 2)$ . The points  $z_2, w_2$  are the values which  $f(z)$  approaches as  $z \rightarrow 0$  from opposite sides of  $\partial D = \gamma$ .

Next theorems on quasiconformal mappings due to Ahlfors [1] and Teichmüller [8] imply the existence of a positive constant  $\delta_2 = \delta_2(a) \leq \delta_1$  such that  $|z_2| \leq \frac{1}{5}$  and  $|w_2| \leq \frac{1}{5}$  whenever  $\|S_f\|_D \leq \delta_2$ .

Finally set  $\delta = \delta_2$ . If  $\|S_f\|_D \leq \delta$ , then

$$|z_2 - w_2| \leq \frac{2}{5} < \frac{4}{5b} \leq \frac{1}{b} \min(|1 - z_2|, |-1 - w_2|),$$

Lemma 2 implies that  $z_2 = w_2$  and hence  $f(D)$  is not a Jordan domain. A complete proof for Theorem 5 is given in [5].

#### 5. CONCLUDING REMARKS

We have obtained Theorems 1 and 3 from the stronger conclusion in Theorem 5. We conclude by stating a result for multiply connected domains which implies Theorems 2 and 4.

Given a function  $\varphi$  defined in an arbitrary proper subdomain  $D$  of  $\mathbb{C}$ , we introduce the norm

$$\|\varphi\|_D^* = \sup_{z \in D} |\varphi(z)| \operatorname{dist}(z, \partial D)^2.$$

When  $D$  is simply connected, classical estimates due to Koebe and Schwarz imply that

$$\frac{1}{4} \operatorname{dist}(z, \partial D)^{-1} \leq \rho_D(z) \leq \operatorname{dist}(z, \partial D)^{-1}$$

for  $z \in D$ , and hence that

$$\|\varphi\|_D^* \leq \|\varphi\|_D \leq 16 \|\varphi\|_D^*.$$

Theorem 6 in [4] and a recent result due to B. Osgood [7] yield the following extension of Theorem 4.

**THEOREM 6.** *A finitely connected proper subdomain  $D$  of  $\mathbf{C}$  is bounded by quasiconformal circles or points if and only if there exists a positive constant  $\delta$  such that  $f$  is univalent in  $D$  whenever  $f$  is meromorphic in  $D$  with  $\|S_f\|_D^* \leq \delta$ .*

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