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4. OUTLINE OF THE PROOF OF THEOREM 5

Fix $a \in \left(0, \frac{1}{8\pi}\right)$ and let $D = \overline{\mathbb{C}} - \gamma$, where

$$\gamma = \{z = \pm i e^{(-a+i)t} : 0 \leq t < \infty\} \cup \{0\}.$$

Then D is a simply connected domain which contains the disjoint $e^{2\pi a}$ -spirals

$$\alpha = \{z = e^{(-a+i)t} : 0 < t < \infty\}, \quad \beta = \{z : -z \in \alpha\}.$$

Next let f denote any conformal mapping of D which fixes the points $1, -1, \infty$. To complete the proof of Theorem 5 it is sufficient to show that there exists a positive constant $\delta = \delta(a)$ such that $f(D)$ is not a Jordan domain whenever $\|S_f\|_D \leq \delta$. This is done in three steps.

First using Lemma 1 and a normal family type argument, we can prove that there exists a $\delta_1 = \delta_1(a) > 0$ with the following property. If $\|S_f\|_D \leq \delta_1$, then $f(\alpha)$ and $f(\beta)$ are b -spirals from 1 onto z_2 and from -1 onto w_2 , respectively, where $b \in (1, 2)$. The points z_2, w_2 are the values which $f(z)$ approaches as $z \rightarrow 0$ from opposite sides of $\partial D = \gamma$.

Next theorems on quasiconformal mappings due to Ahlfors [1] and Teichmüller [8] imply the existence of a positive constant $\delta_2 = \delta_2(a) \leq \delta_1$ such that $|z_2| \leq \frac{1}{5}$ and $|w_2| \leq \frac{1}{5}$ whenever $\|S_f\|_D \leq \delta_2$.

Finally set $\delta = \delta_2$. If $\|S_f\|_D \leq \delta$, then

$$|z_2 - w_2| \leq \frac{2}{5} < \frac{4}{5b} \leq \frac{1}{b} \min(|1 - z_2|, |-1 - w_2|),$$

Lemma 2 implies that $z_2 = w_2$ and hence $f(D)$ is not a Jordan domain. A complete proof for Theorem 5 is given in [5].

5. CONCLUDING REMARKS

We have obtained Theorems 1 and 3 from the stronger conclusion in Theorem 5. We conclude by stating a result for multiply connected domains which implies Theorems 2 and 4.

Given a function φ defined in an arbitrary proper subdomain D of \mathbb{C} , we introduce the norm

$$\|\varphi\|_D^* = \sup_{z \in D} |\varphi(z)| \operatorname{dist}(z, \partial D)^2.$$

When D is simply connected, classical estimates due to Koebe and Schwarz imply that

$$\frac{1}{4} \operatorname{dist}(z, \partial D)^{-1} \leq \rho_D(z) \leq \operatorname{dist}(z, \partial D)^{-1}$$

for $z \in D$, and hence that

$$\|\varphi\|_D^* \leq \|\varphi\|_D \leq 16 \|\varphi\|_D^*.$$

Theorem 6 in [4] and a recent result due to B. Osgood [7] yield the following extension of Theorem 4.

THEOREM 6. *A finitely connected proper subdomain D of \mathbf{C} is bounded by quasiconformal circles or points if and only if there exists a positive constant δ such that f is univalent in D whenever f is meromorphic in D with $\|S_f\|_D^* \leq \delta$.*

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