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We give an argument to show the equivalence of Theorems 1 and 3. Suppose first that Theorem 1 holds. Then there exists a $\varphi \in S$ and a $\delta > 0$ such that $\|\psi - \varphi\| > \delta$ for all $\psi \in T$. Choose g conformal in L with $S_g = \varphi$, let $D = g(L)$ and suppose that f is conformal in D with $\|S_f\|_D \leq \delta$. Then $h = f \circ g$ is conformal in L ,

$$(2) \quad S_h = (S_{f \circ g})(g')^2 + S_g$$

by the composition law for the Schwarzian derivative, and hence $\psi = S_h \in S$ with

$$\|\psi - \varphi\| = \|S_h - S_g\|_L = \|S_f\|_D \leq \delta.$$

Thus $\psi \notin T$, h does not have a quasiconformal extension to $\overline{\mathbf{C}}$, and $\partial f(D) = \partial h(L)$ is not a quasiconformal circle. Hence Theorem 3 holds.

Suppose next that Theorem 3 holds, let $\varphi = S_g$ where g is any conformal mapping of L onto D , and choose any $\psi \in S$ with $\|\psi - \varphi\| \leq \delta$. Then $\psi = S_h$ where h is conformal in L , $f = h \circ g^{-1}$ is conformal in D and from (2) we obtain

$$\|S_f\|_D = \|S_h - S_g\|_L = \|\psi - \varphi\| \leq \delta.$$

Hence $\partial h(L) = \partial f(D)$ is not a quasiconformal circle, h does not have a quasiconformal extension to $\overline{\mathbf{C}}$ and $\psi \notin T$. Thus the distance from φ to T is at least δ and Theorem 1 holds.

A simple modification of the above argument yields the equivalence of Theorems 2 and 4.

Theorems 1 and 3 are immediate consequences of the following result.

THEOREM 5. *There exists a simply connected domain D and a positive constant δ such that $f(D)$ is not a Jordan domain whenever f is conformal in D with $\|S_f\|_D \leq \delta$.*

3. SPIRALS

The proof of Theorem 5 is based on two results for a class of spirals.

DEFINITION. *We say that an open arc α in \mathbf{C} is a b -spiral from z_1 onto z_2 if α has the representation*

$$z = (z_1 - z_2)r(t)e^{it} + z_2, \quad 0 < t < \infty,$$

where $r(t)$ is positive and continuous with

$$\lim_{t \rightarrow 0} r(t) = 1, \quad \lim_{t \rightarrow \infty} r(t) = 0,$$

and where $r(t_1) \leq b r(t_2)$ for all t_1, t_2 with $|t_1 - t_2| \leq 2\pi$.

When a is a positive constant, the arc

$$\alpha = \{z = e^{(-a+i)t} : 0 < t < \infty\}$$

is an $e^{2\pi a}$ -spiral from 1 onto 0. Moreover,

$$(3) \quad k(z)|z| = c, \quad \frac{d}{ds}(z)|z|^2 = d$$

for all $z \in \alpha$, where c and d are positive constants with $d = ac^2$, and where k and s denote the curvature and arclength of α .

The first result we need shows that a curvature condition, similar to (3), is sufficient to guarantee that an open arc is a b -spiral.

LEMMA 1. Suppose that α is an analytic open arc with 1 and 0 as endpoints, and suppose that

$$(4) \quad c_1 \leq k(z)|z| \leq c_2, \quad d_1 \leq \frac{d}{ds}(z)|z|^2 \leq d_2$$

for all $z \in \alpha$, where c_1, c_2, d_1, d_2 are positive constants with $4\pi d_2 < c_1^2$. Then α is a rectifiable b -spiral from 1 onto 0 where

$$b = \frac{c_1 c_2}{c_1^2 - 4\pi d_2}.$$

The second result we require implies that when b is near 1, the points onto which two disjoint b -spirals converge either coincide or are separated by a distance greater than $\frac{1}{2b^2}$ times the diameter of the smaller spiral.

LEMMA 2. Suppose that α and β are disjoint b -spirals from z_1 onto z_2 and from w_1 onto w_2 , respectively. If $b \in (1, 2)$, then either $z_2 = w_2$ or

$$|z_2 - w_2| > \frac{1}{b} \min(|z_1 - z_2|, |w_1 - w_2|).$$