Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 24 (1978)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: REMARKS ON THE UNIVERS AL TEICHMÜLLER SPACE
Autor: Gehring, F. W.
Kapitel: 2. Reformulations in the plane
DOI: https://doi.org/10.5169/seals-49698

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 24.01.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

The set T is called the universal Teichmüller space. An important result due to Ahlfors and Bers shows that each Teichmüller space of a Riemann surface R or of a Fuchsian group G has a canonical embedding in the space T. See, for example, [3].

It is natural to ask if there exist relations, other than (1), between S and T as subsets of B_2. Compactness results for conformal mappings show that S is closed in B_2. Hence Bers asked in [2] and [3] if one can characterize S in terms of T as follows.

QUESTION. *Is S the closure of T?*

We shall answer this question in the negative by sketching a proof for the following result.

THEOREM 1. *There exists a φ in S which does not lie in the closure of T.*

On the other hand, we have the following characterization of T in terms of S. See [4].

THEOREM 2. *T is the interior of S.*

2. REFORMULATIONS IN THE PLANE

A set $E \subset \overline{\mathbf{C}}$ is said to be a *quasiconformal circle* if there exists a quasiconformal mapping f defined in $\overline{\mathbf{C}}$ which maps the unit circle $\{z: |z| = 1\}$ onto E.

Theorems 1 and 2 are then respectively equivalent to the following two results on plane domains D.

THEOREM 3. *There exists a simply connected domain D and a positive constant δ such that $f(D)$ is not bounded by a quasiconformal circle whenever f is conformal in D with $\| S_f \|_D \leqslant \delta$.*

THEOREM 4. *A simply connected domain D is bounded by a quasiconformal circle if and only if there exists a positive constant δ such that f is univalent in D whenever f is meromorphic in D with $\| S_f \|_D \leqslant \delta$.*

We give an argument to show the equivalence of Theorems 1 and 3. Suppose first that Theorem 1 holds. Then there exists a $\varphi \in S$ and a $\delta > 0$ such that $\|\psi - \varphi\| > \delta$ for all $\psi \in T$. Choose g conformal in L with $S_g = \varphi$, let $D = g(L)$ and suppose that f is conformal in D with $\|S_f\|_D \leqslant \delta$. Then $h = f \circ g$ is conformal in L,

$$S_h = (S_{f \circ g})(g')^2 + S_g \tag{2}$$

by the composition law for the Schwarzian derivative, and hence $\psi = S_h \in S$ with

$$\|\psi - \varphi\| = \|S_h - S_g\|_L = \|S_f\|_D \leqslant \delta .$$

Thus $\psi \notin T$, h does not have a quasiconformal extension to $\overline{\mathbf{C}}$, and $\partial f(D) = \partial h(L)$ is not a quasiconformal circle. Hence Theorem 3 holds.

Suppose next that Theorem 3 holds, let $\varphi = S_g$ where g is any conformal mapping of L onto D, and choose any $\psi \in S$ with $\|\psi - \varphi\| \leqslant \delta$. Then $\psi = S_h$ where h is conformal in L, $f = h \circ g^{-1}$ is conformal in D and from (2) we obtain

$$\|S_f\|_D = \|S_h - S_g\|_L = \|\psi - \varphi\| \leqslant \delta .$$

Hence $\partial h(L) = \partial f(D)$ is not a quasiconformal circle, h does not have a quasiconformal extension to $\overline{\mathbf{C}}$ and $\psi \notin T$. Thus the distance from φ to T is at least δ and Theorem 1 holds.

A simple modification of the above argument yields the equivalence of Theorems 2 and 4.

Theorems 1 and 3 are immediate consequences of the following result.

THEOREM 5. *There exists a simply connected domain D and a positive constant δ such that $f(D)$ is not a Jordan domain whenever f is conformal in D with $\|S_f\|_D \leqslant \delta$.*

3. SPIRALS

The proof of Theorem 5 is based on two results for a class of spirals.

DEFINITION. *We say that an open arc α in $\mathbf{C}$ is a b-spiral from z_1 onto z_2 if α has the representation*

$$z = (z_1 - z_2)\, r(t)\, e^{it} + z_2 , \quad 0 < t < \infty ,$$

where $r(t)$ is positive and continuous with