Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 24 (1978)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: REMARKS ON THE UNIVERS AL TEICHMÜLLER SPACE

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Kapitel: 2. Reformulations in the plane

DOI: https://doi.org/10.5169/seals-49698

The set T is called the universal Teichmüller space. An important result due to Ahlfors and Bers shows that each Teichmüller space of a Riemann surface R or of a Fuchsian group G has a canonical embedding in the space T. See, for example, [3].

It is natural to ask if there exist relations, other than (1), between S and T as subsets of B_2. Compactness results for conformal mappings show that S is closed in B_2. Hence Bers asked in [2] and [3] if one can characterize S in terms of T as follows.

QUESTION. *Is S the closure of T?*

We shall answer this question in the negative by sketching a proof for the following result.

THEOREM 1. *There exists a φ in S which does not lie in the closure of T.*

On the other hand, we have the following characterization of T in terms of S. See [4].

THEOREM 2. *T is the interior of S.*

2. Reformulations in the plane

A set $E \subset \overline{\mathbf{C}}$ is said to be a *quasiconformal circle* if there exists a quasiconformal mapping f defined in $\overline{\mathbf{C}}$ which maps the unit circle $\{z: |z| = 1\}$ onto E.

Theorems 1 and 2 are then respectively equivalent to the following two results on plane domains D.

THEOREM 3. *There exists a simply connected domain D and a positive constant δ such that $f(D)$ is not bounded by a quasiconformal circle whenever f is conformal in D with $\| S_f \|_D \leqslant \delta$.*

THEOREM 4. *A simply connected domain D is bounded by a quasiconformal circle if and only if there exists a positive constant δ such that f is univalent in D whenever f is meromorphic in D with $\| S_f \|_D \leqslant \delta$.*

We give an argument to show the equivalence of Theorems 1 and 3. Suppose first that Theorem 1 holds. Then there exists a $\varphi \in S$ and a $\delta > 0$ such that $\| \psi - \varphi \| > \delta$ for all $\psi \in T$. Choose g conformal in L with $S_g = \varphi$, let $D = g(L)$ and suppose that f is conformal in D with $\| S_f \|_D \leqslant \delta$. Then $h = f \circ g$ is conformal in L,

$$S_h = (S_{f} \circ g)(g')^2 + S_g \tag{2}$$

by the composition law for the Schwarzian derivative, and hence $\psi = S_h \in S$ with

$$\| \psi - \varphi \| = \| S_h - S_g \|_L = \| S_f \|_D \leqslant \delta .$$

Thus $\psi \notin T$, h does not have a quasiconformal extension to $\overline{\mathbf{C}}$, and $\partial f(D) = \partial h(L)$ is not a quasiconformal circle. Hence Theorem 3 holds.

Suppose next that Theorem 3 holds, let $\varphi = S_g$ where g is any conformal mapping of L onto D, and choose any $\psi \in S$ with $\| \psi - \varphi \| \leqslant \delta$. Then $\psi = S_h$ where h is conformal in L, $f = h \circ g^{-1}$ is conformal in D and from (2) we obtain

$$\| S_f \|_D = \| S_h - S_g \|_L = \| \psi - \varphi \| \leqslant \delta .$$

Hence $\partial h(L) = \partial f(D)$ is not a quasiconformal circle, h does not have a quasiconformal extension to $\overline{\mathbf{C}}$ and $\psi \notin T$. Thus the distance from φ to T is at least δ and Theorem 1 holds.

A simple modification of the above argument yields the equivalence of Theorems 2 and 4.

Theorems 1 and 3 are immediate consequences of the following result.

THEOREM 5. *There exists a simply connected domain D and a positive constant δ such that $f(D)$ is not a Jordan domain whenever f is conformal in D with $\| S_f \|_D \leqslant \delta$.*

3. SPIRALS

The proof of Theorem 5 is based on two results for a class of spirals.

DEFINITION. *We say that an open arc α in $\mathbf{C}$ is a b-spiral from z_1 onto z_2 if α has the representation*

$$z = (z_1 - z_2)\, r(t)\, e^{it} + z_2 , \quad 0 < t < \infty ,$$

where $r(t)$ is positive and continuous with