

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 24 (1978)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: REMARKS ON THE UNIVERSAL TEICHMÜLLER SPACE
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Kapitel: 1. Introduction
DOI: <https://doi.org/10.5169/seals-49698>

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REMARKS ON THE UNIVERSAL TEICHMÜLLER SPACE¹

by F. W. GEHRING²

1. INTRODUCTION

Suppose that D is a simply connected domain of hyperbolic type in the extended complex plane $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Then the hyperbolic or noneuclidean metric ρ_D in D is given by

$$\rho_D(z) = (1 - |g(z)|^2)^{-1} |g'(z)|,$$

where g is any conformal mapping of D onto the unit disk $\{z: |z| < 1\}$. For each function φ defined in D we introduce the norm

$$\|\varphi\|_D = \sup_{z \in D} |\varphi(z)| \rho_D(z)^{-2}.$$

Next for each function f which is meromorphic and locally univalent in D we let S_f denote the Schwarzian derivative of f . At finite points of D which are not poles of f , S_f is given by

$$S_f = \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2,$$

and the definition is extended to ∞ and the poles of f by means of inversion.

Now let L denote the lower half plane $\{z = x + iy: y < 0\}$ and let $B_2 = B_2(L, 1)$ denote the complex Banach space of functions φ analytic in L with the norm

$$\|\varphi\| = \|\varphi\|_L = \sup_{z \in L} 4y^2 |\varphi(z)| < \infty.$$

Next let S denote the family of functions $\varphi = S_g$ where g is conformal in L , and let $T = T(1)$ denote the subfamily of those $\varphi = S_g$ where g has a quasiconformal extension to $\bar{\mathbf{C}}$. From [6] it follows that $\|\varphi\| \leq 6$ for all $\varphi \in S$, and hence that

$$(1) \quad T \subset S \subset B_2.$$

¹) Communicated to an International Symposium on Analysis, held in honour of Professor Albert Pfluger, ETH Zürich, 1978.

²) This research was supported in part by a grant from the U.S. National Science Foundation, Grant MCS-77-02842.

The set T is called the universal Teichmüller space. An important result due to Ahlfors and Bers shows that each Teichmüller space of a Riemann surface R or of a Fuchsian group G has a canonical embedding in the space T . See, for example, [3].

It is natural to ask if there exist relations, other than (1), between S and T as subsets of B_2 . Compactness results for conformal mappings show that S is closed in B_2 . Hence Bers asked in [2] and [3] if one can characterize S in terms of T as follows.

QUESTION. *Is S the closure of T ?*

We shall answer this question in the negative by sketching a proof for the following result.

THEOREM 1. *There exists a φ in S which does not lie in the closure of T .*

On the other hand, we have the following characterization of T in terms of S . See [4].

THEOREM 2. *T is the interior of S .*

2. REFORMULATIONS IN THE PLANE

A set $E \subset \overline{\mathbb{C}}$ is said to be a *quasiconformal circle* if there exists a quasiconformal mapping f defined in $\overline{\mathbb{C}}$ which maps the unit circle $\{z: |z| = 1\}$ onto E .

Theorems 1 and 2 are then respectively equivalent to the following two results on plane domains D .

THEOREM 3. *There exists a simply connected domain D and a positive constant δ such that $f(D)$ is not bounded by a quasiconformal circle whenever f is conformal in D with $\|S_f\|_D \leq \delta$.*

THEOREM 4. *A simply connected domain D is bounded by a quasiconformal circle if and only if there exists a positive constant δ such that f is univalent in D whenever f is meromorphic in D with $\|S_f\|_D \leq \delta$.*