# REMARKS ON THE UNIVERSAL TEICHMÜLLER SPACE[1]

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## 1. INTRODUCTION

Suppose that $D$ is a simply connected domain of hyperbolic type in the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Then the hyperbolic or noneuclidean metric $\rho_D$ in $D$ is given by

$$\rho_D(z) = (1 - |g(z)|^2)^{-1} |g'(z)|,$$

where $g$ is any conformal mapping of $D$ onto the unit disk $\{z : |z| < 1\}$. For each function $\varphi$ defined in $D$ we introduce the norm

$$\|\varphi\|_D = \sup_{z \in D} |\varphi(z)| \rho_D(z)^{-2}.$$

Next for each function $f$ which is meromorphic and locally univalent in $D$ we let $S_f$ denote the Schwarzian derivative of $f$. At finite points of $D$ which are not poles of $f$, $S_f$ is given by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2,$$

and the definition is extended to $\infty$ and the poles of $f$ by means of inversion.

Now let $L$ denote the lower half plane $\{z = x + iy : y < 0\}$ and let $B_2 = B_2(L, 1)$ denote the complex Banach space of functions $\varphi$ analytic in $L$ with the norm

$$\|\varphi\| = \|\varphi\|_L = \sup_{z \in L} 4y^2 |\varphi(z)| < \infty.$$

Next let $S$ denote the family of functions $\varphi = S_g$ where $g$ is conformal in $L$, and let $T = T(1)$ denote the subfamily of those $\varphi = S_g$ where $g$ has a quasiconformal extension to $\overline{\mathbf{C}}$. From [6] it follows that $\|\varphi\| \leqslant 6$ for all $\varphi \in S$, and hence that

$$T \subset S \subset B_2. \tag{1}$$

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The set $T$ is called the universal Teichmüller space. An important result due to Ahlfors and Bers shows that each Teichmüller space of a Riemann surface $R$ or of a Fuchsian group $G$ has a canonical embedding in the space $T$. See, for example, [3].

It is natural to ask if there exist relations, other than (1), between $S$ and $T$ as subsets of $B_2$. Compactness results for conformal mappings show that $S$ is closed in $B_2$. Hence Bers asked in [2] and [3] if one can characterize $S$ in terms of $T$ as follows.

QUESTION. *Is $S$ the closure of $T$?*

We shall answer this question in the negative by sketching a proof for the following result.

THEOREM 1. *There exists a $\varphi$ in $S$ which does not lie in the closure of $T$.*

On the other hand, we have the following characterization of $T$ in terms of $S$. See [4].

THEOREM 2. *$T$ is the interior of $S$.*

## 2. Reformulations in the plane

A set $E \subset \overline{\mathbf{C}}$ is said to be a *quasiconformal circle* if there exists a quasiconformal mapping $f$ defined in $\overline{\mathbf{C}}$ which maps the unit circle $\{z: |z| = 1\}$ onto $E$.

Theorems 1 and 2 are then respectively equivalent to the following two results on plane domains $D$.

THEOREM 3. *There exists a simply connected domain $D$ and a positive constant $\delta$ such that $f(D)$ is not bounded by a quasiconformal circle whenever $f$ is conformal in $D$ with $\| S_f \|_D \leqslant \delta$.*

THEOREM 4. *A simply connected domain $D$ is bounded by a quasiconformal circle if and only if there exists a positive constant $\delta$ such that $f$ is univalent in $D$ whenever $f$ is meromorphic in $D$ with $\| S_f \|_D \leqslant \delta$.*