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# REMARKS ON THE UNIVERSAL TEICHMÜLLER SPACE<sup>1</sup>

by F. W. GEHRING<sup>2</sup>

## 1. INTRODUCTION

Suppose that  $D$  is a simply connected domain of hyperbolic type in the extended complex plane  $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ . Then the hyperbolic or noneuclidean metric  $\rho_D$  in  $D$  is given by

$$\rho_D(z) = (1 - |g(z)|^2)^{-1} |g'(z)|,$$

where  $g$  is any conformal mapping of  $D$  onto the unit disk  $\{z: |z| < 1\}$ . For each function  $\varphi$  defined in  $D$  we introduce the norm

$$\|\varphi\|_D = \sup_{z \in D} |\varphi(z)| \rho_D(z)^{-2}.$$

Next for each function  $f$  which is meromorphic and locally univalent in  $D$  we let  $S_f$  denote the Schwarzian derivative of  $f$ . At finite points of  $D$  which are not poles of  $f$ ,  $S_f$  is given by

$$S_f = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2,$$

and the definition is extended to  $\infty$  and the poles of  $f$  by means of inversion.

Now let  $L$  denote the lower half plane  $\{z = x + iy: y < 0\}$  and let  $B_2 = B_2(L, 1)$  denote the complex Banach space of functions  $\varphi$  analytic in  $L$  with the norm

$$\|\varphi\| = \|\varphi\|_L = \sup_{z \in L} 4y^2 |\varphi(z)| < \infty.$$

Next let  $S$  denote the family of functions  $\varphi = S_g$  where  $g$  is conformal in  $L$ , and let  $T = T(1)$  denote the subfamily of those  $\varphi = S_g$  where  $g$  has a quasiconformal extension to  $\bar{\mathbf{C}}$ . From [6] it follows that  $\|\varphi\| \leq 6$  for all  $\varphi \in S$ , and hence that

$$(1) \quad T \subset S \subset B_2.$$

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The set  $T$  is called the universal Teichmüller space. An important result due to Ahlfors and Bers shows that each Teichmüller space of a Riemann surface  $R$  or of a Fuchsian group  $G$  has a canonical embedding in the space  $T$ . See, for example, [3].

It is natural to ask if there exist relations, other than (1), between  $S$  and  $T$  as subsets of  $B_2$ . Compactness results for conformal mappings show that  $S$  is closed in  $B_2$ . Hence Bers asked in [2] and [3] if one can characterize  $S$  in terms of  $T$  as follows.

QUESTION. *Is  $S$  the closure of  $T$ ?*

We shall answer this question in the negative by sketching a proof for the following result.

THEOREM 1. *There exists a  $\varphi$  in  $S$  which does not lie in the closure of  $T$ .*

On the other hand, we have the following characterization of  $T$  in terms of  $S$ . See [4].

THEOREM 2.  *$T$  is the interior of  $S$ .*

## 2. REFORMULATIONS IN THE PLANE

A set  $E \subset \overline{\mathbb{C}}$  is said to be a *quasiconformal circle* if there exists a quasiconformal mapping  $f$  defined in  $\overline{\mathbb{C}}$  which maps the unit circle  $\{z: |z| = 1\}$  onto  $E$ .

Theorems 1 and 2 are then respectively equivalent to the following two results on plane domains  $D$ .

THEOREM 3. *There exists a simply connected domain  $D$  and a positive constant  $\delta$  such that  $f(D)$  is not bounded by a quasiconformal circle whenever  $f$  is conformal in  $D$  with  $\|S_f\|_D \leq \delta$ .*

THEOREM 4. *A simply connected domain  $D$  is bounded by a quasiconformal circle if and only if there exists a positive constant  $\delta$  such that  $f$  is univalent in  $D$  whenever  $f$  is meromorphic in  $D$  with  $\|S_f\|_D \leq \delta$ .*

We give an argument to show the equivalence of Theorems 1 and 3. Suppose first that Theorem 1 holds. Then there exists a  $\varphi \in S$  and a  $\delta > 0$  such that  $\|\psi - \varphi\| > \delta$  for all  $\psi \in T$ . Choose  $g$  conformal in  $L$  with  $S_g = \varphi$ , let  $D = g(L)$  and suppose that  $f$  is conformal in  $D$  with  $\|S_f\|_D \leq \delta$ . Then  $h = f \circ g$  is conformal in  $L$ ,

$$(2) \quad S_h = (S_f \circ g)(g')^2 + S_g$$

by the composition law for the Schwarzian derivative, and hence  $\psi = S_h \in S$  with

$$\|\psi - \varphi\| = \|S_h - S_g\|_L = \|S_f\|_D \leq \delta.$$

Thus  $\psi \notin T$ ,  $h$  does not have a quasiconformal extension to  $\overline{\mathbf{C}}$ , and  $\partial f(D) = \partial h(L)$  is not a quasiconformal circle. Hence Theorem 3 holds.

Suppose next that Theorem 3 holds, let  $\varphi = S_g$  where  $g$  is any conformal mapping of  $L$  onto  $D$ , and choose any  $\psi \in S$  with  $\|\psi - \varphi\| \leq \delta$ . Then  $\psi = S_h$  where  $h$  is conformal in  $L$ ,  $f = h \circ g^{-1}$  is conformal in  $D$  and from (2) we obtain

$$\|S_f\|_D = \|S_h - S_g\|_L = \|\psi - \varphi\| \leq \delta.$$

Hence  $\partial h(L) = \partial f(D)$  is not a quasiconformal circle,  $h$  does not have a quasiconformal extension to  $\overline{\mathbf{C}}$  and  $\psi \notin T$ . Thus the distance from  $\varphi$  to  $T$  is at least  $\delta$  and Theorem 1 holds.

A simple modification of the above argument yields the equivalence of Theorems 2 and 4.

Theorems 1 and 3 are immediate consequences of the following result.

**THEOREM 5.** *There exists a simply connected domain  $D$  and a positive constant  $\delta$  such that  $f(D)$  is not a Jordan domain whenever  $f$  is conformal in  $D$  with  $\|S_f\|_D \leq \delta$ .*

### 3. SPIRALS

The proof of Theorem 5 is based on two results for a class of spirals.

**DEFINITION.** *We say that an open arc  $\alpha$  in  $\mathbf{C}$  is a  $b$ -spiral from  $z_1$  onto  $z_2$  if  $\alpha$  has the representation*

$$z = (z_1 - z_2)r(t)e^{it} + z_2, \quad 0 < t < \infty,$$

where  $r(t)$  is positive and continuous with

$$\lim_{t \rightarrow 0} r(t) = 1, \quad \lim_{t \rightarrow \infty} r(t) = 0,$$

and where  $r(t_1) \leq b r(t_2)$  for all  $t_1, t_2$  with  $|t_1 - t_2| \leq 2\pi$ .

When  $a$  is a positive constant, the arc

$$\alpha = \{z = e^{(-a+i)t} : 0 < t < \infty\}$$

is an  $e^{2\pi a}$ -spiral from 1 onto 0. Moreover,

$$(3) \quad k(z) |z| = c, \quad \frac{dk}{ds}(z) |z|^2 = d$$

for all  $z \in \alpha$ , where  $c$  and  $d$  are positive constants with  $d = ac^2$ , and where  $k$  and  $s$  denote the curvature and arclength of  $\alpha$ .

The first result we need shows that a curvature condition, similar to (3), is sufficient to guarantee that an open arc is a  $b$ -spiral.

LEMMA 1. Suppose that  $\alpha$  is an analytic open arc with 1 and 0 as endpoints, and suppose that

$$(4) \quad c_1 \leq k(z) |z| \leq c_2, \quad d_1 \leq \frac{dk}{ds}(z) |z|^2 \leq d_2$$

for all  $z \in \alpha$ , where  $c_1, c_2, d_1, d_2$  are positive constants with  $4\pi d_2 < c_1^2$ . Then  $\alpha$  is a rectifiable  $b$ -spiral from 1 onto 0 where

$$b = \frac{c_1 c_2}{c_1^2 - 4\pi d_2}.$$

The second result we require implies that when  $b$  is near 1, the points onto which two disjoint  $b$ -spirals converge either coincide or are separated by a distance greater than  $\frac{1}{2b^2}$  times the diameter of the smaller spiral.

LEMMA 2. Suppose that  $\alpha$  and  $\beta$  are disjoint  $b$ -spirals from  $z_1$  onto  $z_2$  and from  $w_1$  onto  $w_2$ , respectively. If  $b \in (1, 2)$ , then either  $z_2 = w_2$  or

$$|z_2 - w_2| > \frac{1}{b} \min(|z_1 - z_2|, |w_1 - w_2|).$$

#### 4. OUTLINE OF THE PROOF OF THEOREM 5

Fix  $a \in \left(0, \frac{1}{8\pi}\right)$  and let  $D = \overline{\mathbb{C}} - \gamma$ , where

$$\gamma = \{z = \pm i e^{(-a+i)t} : 0 \leq t < \infty\} \cup \{0\}.$$

Then  $D$  is a simply connected domain which contains the disjoint  $e^{2\pi a}$ -spirals

$$\alpha = \{z = e^{(-a+i)t} : 0 < t < \infty\}, \quad \beta = \{z : -z \in \alpha\}.$$

Next let  $f$  denote any conformal mapping of  $D$  which fixes the points  $1, -1, \infty$ . To complete the proof of Theorem 5 it is sufficient to show that there exists a positive constant  $\delta = \delta(a)$  such that  $f(D)$  is not a Jordan domain whenever  $\|S_f\|_D \leq \delta$ . This is done in three steps.

First using Lemma 1 and a normal family type argument, we can prove that there exists a  $\delta_1 = \delta_1(a) > 0$  with the following property. If  $\|S_f\|_D \leq \delta_1$ , then  $f(\alpha)$  and  $f(\beta)$  are  $b$ -spirals from 1 onto  $z_2$  and from  $-1$  onto  $w_2$ , respectively, where  $b \in (1, 2)$ . The points  $z_2, w_2$  are the values which  $f(z)$  approaches as  $z \rightarrow 0$  from opposite sides of  $\partial D = \gamma$ .

Next theorems on quasiconformal mappings due to Ahlfors [1] and Teichmüller [8] imply the existence of a positive constant  $\delta_2 = \delta_2(a) \leq \delta_1$  such that  $|z_2| \leq \frac{1}{5}$  and  $|w_2| \leq \frac{1}{5}$  whenever  $\|S_f\|_D \leq \delta_2$ .

Finally set  $\delta = \delta_2$ . If  $\|S_f\|_D \leq \delta$ , then

$$|z_2 - w_2| \leq \frac{2}{5} < \frac{4}{5b} \leq \frac{1}{b} \min(|1 - z_2|, |-1 - w_2|),$$

Lemma 2 implies that  $z_2 = w_2$  and hence  $f(D)$  is not a Jordan domain.

A complete proof for Theorem 5 is given in [5].

#### 5. CONCLUDING REMARKS

We have obtained Theorems 1 and 3 from the stronger conclusion in Theorem 5. We conclude by stating a result for multiply connected domains which implies Theorems 2 and 4.

Given a function  $\varphi$  defined in an arbitrary proper subdomain  $D$  of  $\mathbb{C}$ , we introduce the norm

$$\|\varphi\|_D^* = \sup_{z \in D} |\varphi(z)| \operatorname{dist}(z, \partial D)^2.$$

When  $D$  is simply connected, classical estimates due to Koebe and Schwarz imply that

$$\frac{1}{4} \operatorname{dist}(z, \partial D)^{-1} \leq \rho_D(z) \leq \operatorname{dist}(z, \partial D)^{-1}$$

for  $z \in D$ , and hence that

$$\|\varphi\|_D^* \leq \|\varphi\|_D \leq 16 \|\varphi\|_D^*.$$

Theorem 6 in [4] and a recent result due to B. Osgood [7] yield the following extension of Theorem 4.

**THEOREM 6.** *A finitely connected proper subdomain  $D$  of  $\mathbb{C}$  is bounded by quasiconformal circles or points if and only if there exists a positive constant  $\delta$  such that  $f$  is univalent in  $D$  whenever  $f$  is meromorphic in  $D$  with  $\|S_f\|_D^* \leq \delta$ .*

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