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THE LEVI PROBLEM AND PSEUDO-CONVEX DOMAINS: A SURVEY [1]

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§ 1. THE LEVI PROBLEM

One of the classical problems of several complex variables is the *Levi Problem*: the characterisation of domains in $\mathbf{C}^n$ on which there exist holomorphic functions which are singular at every boundary point.

Domains on which such functions exist are called domains of holomorphy. If $n > 1$, there exist domains that are not domains of holomorphy. E. E. Levi found conditions that the boundary of a domain Ω has to satisfy in order that Ω be a domain of holomorphy. The "Levi Problem" has its origin in the question of whether the conditions given by Levi are sufficient to guarantee that Ω is a domain of holomorphy.

A definitive solution of the Levi problem has been known for some 25 years, thanks, principally, to the work of K. Oka. Before stating the result, we introduce some definitions and notation.

A real valued, C^2-function p defined on an open set Ω in $\mathbf{C}^n$ is called plurisubharmonic, if the hermitian form

$$\sum_{\mu,\nu=1}^{n} \frac{\partial^2 p}{\partial z_\mu \, \partial \bar{z}_\nu} \alpha_\mu \bar{\alpha}_\nu$$

is positive semi-definite at every point of Ω. If it is positive *definite*, p is called strongly plurisubharmonic.

These functions may be looked upon as the complex analogues of convex (strictly convex) functions in $\mathbf{R}^n$. A real-valued C^2 function u on $\mathbf{R}^n$ is convex (strictly convex) if and only if the real Hessian

$$\sum_{\mu,\nu=1}^{n} \frac{\partial^2 u}{\partial x_\mu \, \partial x_\nu} \alpha_\mu \alpha_\nu$$

is positive semi-definite (definite).

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[1] Communicated to an International Symposium an Analysis, held in honour of Professor Albert Pfluger, ETH Zürich 1978.

Let X be a complex manifold and D a relatively compact open set on X. Let $a \in \partial D$ (the boundary of D). D is said to be pseudo-convex at a if there exists a neighbourhood U of a in X and a plurisubharmonic function p on U such that

$$(*) \qquad U \cap D = \{x \in U \mid p(x) < 0\}.$$

If U and p can be so chosen that p is *strongly* plurisubharmonic [and that (*) holds], D is said to be strongly pseudo-convex at a.

If D is pseudo-convex [strongly pseudo-convex] at every boundary point, it is called pseudo-convex [strongly pseudo-convex].

Strong pseudo-convexity is closely related to strict convexity in the Euclidean sense. In fact, if D has a smooth boundary, then D is strongly pseudo-convex at $a \in \partial D$ if and only if there is a neighbourhood U of a in X and complex coordinates $z_1, \ldots, z_n$ on U such that $U \cap D$ is strictly convex in the Euclidean sense (relative to the coordinates $z_1, \ldots, z_n$).

A complex manifold X is called a Stein manifold if it can be imbedded holomorphically as a closed complex submanifold of some number space $\mathbf{C}^N$. In this case, we also say that X is Stein.

We can now state the main theorem concerning the Levi problem for domains in $\mathbf{C}^n$.

THEOREM. *Let Ω be an open set in $\mathbf{C}^n$. The following properties of Ω are equivalent.*

i) *Ω is a domain of holomorphy.*

ii) *Ω is a Stein manifold.*

iii) *Ω is an increasing union of a sequence of strongly pseudo-convex domains.*

iv) *There exists a strongly plusrisubharmonic function p on Ω such that, for any $c > 0$, the set*

$$\{x \in \Omega \mid p(x) < c\}$$

is relatively compact in Ω.

v) *Let ω be a (C^∞-) differential form of type (p, q) on Ω. Suppose that $q \geqslant 1$ and that $\bar\partial \omega = 0$. Then, there exists a C^∞-form φ of type $(p, q-1)$ on Ω such that $\bar\partial \varphi = \omega$.*

vi) *Any point $a \in \partial \Omega$ has an open neighbourhood U in $\mathbf{C}^n$ such that $U \cap \Omega$ is Stein.*

There are three essentially different methods known of proving this theorem. The first, Oka's [15], is based on the so-called Heftungslemma which proceeds by setting up suitable integral formulae. The second, due to Grauert [10] deals directly with strongly pseudo-convex domains by using sheaf theory and functional analysis. The third, due to Kohn [12] and Hörmander [11], treats the equation $\bar{\partial} \varphi = \omega$ as an overdetermined system of differential equations.

It is natural to ask if the restriction to domains in $\mathbf{C}^n$ is essential, and if there is an analogous theorem for arbitrary complex manifolds.

The first major achievement is Oka's [15]. Let Ω be an unramified domain over $\mathbf{C}^n$ {i.e. Ω is a complex manifold of dimension n provided with a holomorphic map $\pi : \Omega \to \mathbf{C}^n$ whose jacobian determinant is non-zero at every point}.

Then properties ii), iii), iv) are equivalent, and are also equivalent with the following form of vi):

For any $a \in \mathbf{C}^n$, there exists a neighbourhood U in $\mathbf{C}^n$ such that $\pi^{-1}(U)$ is a Stein manifold.

The second major result, due to Grauert [10], is that ii) and iv) are equivalent for *arbitrary* complex manifolds Ω (there is no *a priori* hypothesis concerning the existence of holomorphic functions on Ω). As for iii), J. E. Fornaess [7, 8] has recently constructed examples that show that an increasing union of Stein manifolds is not always again a Stein manifold, so that ii) and iii) are not equivalent for arbitrary complex manifolds Ω.

The problem of deciding when a given manifold is Stein occurs in many contexts. Perhaps the two most important are the following.

1. *The Levi Problem for ramified domains over* $\mathbf{C}^n$.

Let Ω be a complex manifold of dimension n provided with a holomorphic map $\pi : \Omega \to \mathbf{C}^n$ such that $\pi^{-1}(a)$ is a discrete set for any $a \in \mathbf{C}^n$.

We call $\pi : \Omega \to \mathbf{C}^n$ (or Ω) locally Stein if, for any $a \in \mathbf{C}^n$, there is an open neighbourhood U of a in $\mathbf{C}^n$ such that $\pi^{-1}(U)$ is Stein.

The Levi problem for ramified domains is the following: If Ω is locally Stein, is it Stein?

In his paper [15], Oka referred to the difficulty of this problem, and it has attracted much attention since then. It has recently been solved by Fornaess [8], in the negative: There exist complex manifolds of dimension 2 which are ramified domains over $\mathbf{C}^2$ (having at most 2 sheets) that are locally Stein but are not Stein manifolds. His example is sketched at the end of the section.

This is all the more remarkable in view of the following result: Let $\pi : \Omega \to \mathbf{C}^n$ be a ramified domain and let Ω' be a relatively compact open set in Ω such that $\pi : \Omega' \to \mathbf{C}^n$ is locally Stein. Then Ω' is Stein. {This is obtained by combining results of Elencwajg [5] with a result in [1]}.

It is not known if this latter theorem remains valid if Ω is allowed to have singularities.

For some related problems, see Elencwajg [5] and the references given there.

2. *Serre's Problem.*

Let $\pi : X \to B$ be a holomorphic, locally trivial fibre bundle. Suppose that the base B and the fibre $F = \pi^{-1}(b)$, $b \in B$, are Stein manifolds. Is the total space X also a Stein manifold?

There are several positive results.

1°. K. Stein [20]. Any covering manifold of a Stein manifold is Stein. [The case when the fibre is discrete.]

2°. Y. Matsushima-A.Morimoto [14]. Let $\pi : X \to B$ be a fibration associated to a principal fibration with a connected complex Lie group as structure group. If the base and fibre are Stein, X is Stein.

3°. If the fibre is a strongly pseudo-convex domain in $\mathbf{C}^n$, then X is Stein (Fischer [6]; see also Pflug [16] and Stehlé [19]).

4°. Y.-T. Siu [17]. If the fibre is a bounded domain of holomorphy in $\mathbf{C}^n$ whose first Betti number is 0, then X is Stein.

It turns out, however, that the solution, in general, is negative. H. Skoda [18] has constructed an example of a locally trivial fibration $\pi : X \to B$ in which B is a (bounded) domain in $\mathbf{C}$, the fibre is $\mathbf{C}^2$, but the only holomorphic functions on X are those that come from B. This example has been refined and improved by J.-P. Demailly (Sém. Lelong 1976/77).

Here again, as in the Levi problem for ramified domains, relatively compact open sets behave differently: if $\pi : X \to B$ is as above, and D is a relatively compact open set in X which is locally Stein, then D is Stein (Elencwajg [5]).

The Example of Fornaess.

We shall now sketch the idea underlying Fornaess' example mentioned above.

Let $D = \{z \in \mathbf{C} \mid |z| < 1\}$ be the unit disc in $\mathbf{C}$. Set

$$u(z) = \sum_{n \geq 2} \frac{1}{k_n} \log \left(\frac{1}{2} \left| z - \frac{1}{n} \right| \right),$$

where the k_n are integers > 0 increasing rapidly with n. Then u is subharmonic on D, $u < 0$, and u is bounded below on $D - \bigcup_{n \geq 2} D'_n$, where D'_n is a small disc about $\frac{1}{n}$. Moreover

$$u(z) - k_n^{-1} \log \left(\frac{1}{2} \left| z - \frac{1}{n} \right| \right)$$

is continuous at $z = \frac{1}{n}$.

This function u can be modified to yield a function p with the following properties:

p is subharmonic on D, $p < 0$; also there exist small discs D_n about $1/n$ such that p is bounded below on $D - \bigcup_{n \geq 2} D_n$ and

$$p(z) = k_n^{-1} \log \left(\frac{1}{2} \left| z - \frac{1}{n} \right| \right) - 1$$

on D_n.

Although this modification is not essential, we shall suppose, for simplicity of exposition, that this has been done.

Let

$$\Omega = \{(z, w) \in \mathbf{C}^2 \mid w \neq 0, \ |z| < 1, \ p(z) - \log |w| < 0\}.$$

Ω is a domain of holomorphy, and can be represented schematically as follows:

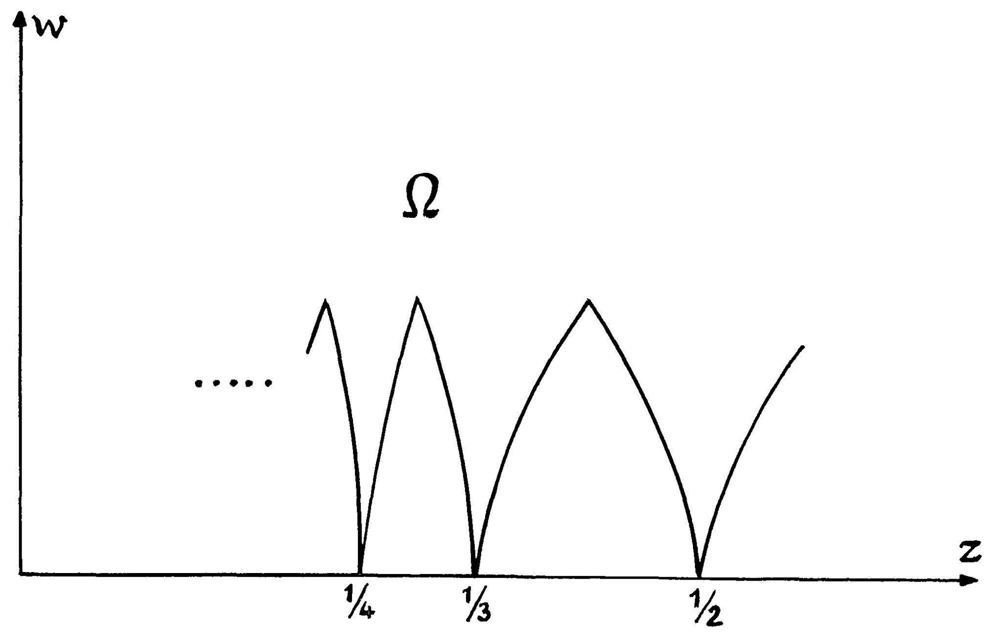

Figure 1

For $z \in D_n$, Ω is given by

$$\{(z, w) \mid |z| < 1, \ |z - 1/n| < 2e^{k_n} |n^{k_n}|\}.$$

Let

$$U_n = \{(\zeta, w) \mid \ |\zeta| < 2e^{k_n}\}, \ U_n' = U_n - \{w = 0\}.$$

Then $\Omega \cap \{D_n \times \mathbf{C}\}$ is the image of U_n' under the map

$$\varphi_n(\zeta, w) = (\frac{1}{n} + \zeta w^{k_n}, w).$$

Define $\psi_n : U_n \to \mathbf{C}^2$ by

$$\psi_n(\zeta, w) = (\frac{1}{n} + \lambda_n \zeta w^{\kappa_n} + \varepsilon_n \zeta^2, w)$$

where κ_n is a large integer, $\lambda_n > 0$ is large, and $\varepsilon_n > 0$ is small (chosen in that order).

For all large κ_n, λ_n, the intersection with $(\mathbf{C} - D_n) \times \mathbf{C}$ of the closure of $\varphi_n(U_n')$ is contained in $\psi_n(U_n)$ for all small ε_n. Further, given a small disc Δ_n of radius ρ_n around $1/n$, ψ_n is injective outside $\psi_n^{-1}(\Delta_n \times \mathbf{C})$ for all small ε_n (if κ_n, λ_n are fixed). Let A_n be the annulus

$$A_n = \{r_n \leqslant |z - \frac{1}{n}| \leqslant R_n\} \ (\rho_n < r_n < R_n < \text{radius}(D_n)).$$

Then for all large λ_n (and small enough ε_n), there is a neighbourhood of $A_n \times \mathbf{C}$ such that its intersection with the closure of $\psi_n(U_n)$ is contained in $\varphi_n(U_n')$.

Schematically, this relationship can be indicated as follows:

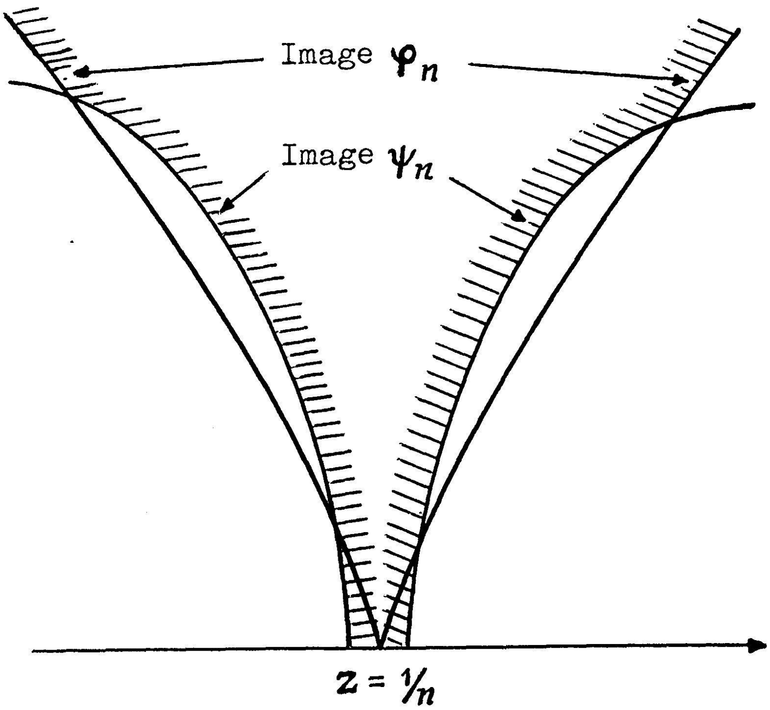

Figure 2

Since ψ_n and φ_n are both injective outside $\Delta_n \times \mathbf{C}$, we can replace Ω by $\Omega \cap (U_n, \psi_n)$ over $(D_n - \{|z| < r_n) \times \mathbf{C}$, (the intersection being relative to ψ_n) and by (U_n, ψ_n) over $\{|z| < r_n\} \times \mathbf{C}$. [See the shaded figure in the diagram above.]

This gives us a manifold X, and a map $\pi : X \to \mathbf{C}^2$, such that $\pi^{-1}(a)$ contains at most two points for any $a \in \mathbf{C}^2$. This is locally Stein; in fact the boundary of $\pi^{-1}(D_n \times \mathbf{C})$ is locally described by an inequality $\max(u, v) < 0$ where u, v are strongly plurisubharmonic. This is known to be sufficient to guarantee that $\pi^{-1}(D_n \times \mathbf{C})$ is Stein. Over a small neighbourhood U of $(0, 0)$, $\pi^{-1}(U)$ is isomorphic to the disjoint union $\bigcup_{n \geq 2} \psi_n^{-1}(U)$. Thus $\pi : X \to \mathbf{C}^2$ is locally Stein. However, X is not Stein. In fact, if

$$K = \{(z, w) \in \mathbf{C}^2 \mid z \text{ real}, \quad 0 \leqslant z \leqslant 1/2, \quad |w| = 1\},$$

the envelope $L = (\pi^{-1}(K))^{\wedge}$ of the compact set $\pi^{-1}(K)$ has the property that $\pi(L)$ contains the discs

$$z = \frac{1}{n}, \ |w| \leqslant 1$$

for $n \geqslant 2$, so that L cannot be compact.

§ 2. Pseudo-convex Domains

Strongly pseudo-convex domains with smooth boundary in $\mathbf{C}^n$ have some very useful properties not shared by arbitrary bounded domains of holomorphy. Here are two such properties.

I. Let Ω be a strongly pseudo-convex domain with smooth boundary. Then Ω has a fundamental system of pseudo-convex neighbourhoods.

II. *Subelliptic estimates.*

We begin with a definition.

If $f \in L^2(\mathbf{R}^n)$, let $\hat{f}$ denote its Fourier transform, and, for a real $s \geqslant 0$, define $\|f\|_s$ by

$$\|f\|_s^2 = \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi$$

We set

$$\mathscr{H}^s(\mathbf{R}^n) = \{f \in L^2(\mathbf{R}^n) \mid \|f\|_s < \infty\}.$$