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To have a model for the homomorphism induced by the inclusion of  $\mathfrak{a}_{n,p}$  in  $\mathfrak{a}_n$ , we have the commutative diagramm

$$\begin{array}{ccc} C^*(\mathfrak{a}_n) & \longrightarrow & C^*(\mathfrak{a}_{n,p}) \\ \uparrow & & \uparrow \\ WU_n & \longleftarrow & WU_{n,p} \end{array}$$

where the second horizontal map sends  $h_i$  on  $h_i' + h_i''$  and  $c_i$  on  $c_i' + c_i''$  (by convention,  $h_i'$  or  $h_i''$  is zero for  $i > p$  or  $i > n-p$ , idem for  $c_i'$  and  $c_i''$ ). Note that the natural map of theorem 1 should map the  $c_i'$  s and  $c_i''$  not on the usual Chern classes defined by the connection but on the polynomials in Chern classes corresponding to  $\sum x_k^i$ , the Chern classes being the elementary symmetric functions in the formal variables  $x_k$ . These horizontal maps are also models for an inclusion of  $F_{n,p}$  in  $F_n$ .

We consider again the bundle  $E$  over  $M$  associated to the tangent bundle of  $M$  and with fiber  $F_n$ . Its restriction above  $N$  contains a subbundle  $E'$  with fiber  $F_{n,p}$ .

**THEOREM.**  $C^*(L_{M,N})$  is a model for the space  $\Gamma_{M,N}$  of continuous sections of the bundle  $E$  whose restriction to  $N$  have values in the subbundle  $E'$ .

To make explicit computations, we construct a model for  $\Gamma_{M,N}$ , which will be finite dimensional in each degree when  $M$  and  $N$  have finite dimensional models. This is the purpose of the next paragraph.

## 9. CONSTRUCTION OF A MODEL FOR $C^*(L_{M,N})$

Consider the commutative diagramm of Lie algebras

$$\begin{array}{ccc} L_{M,N} & \longrightarrow & L_M \\ \downarrow & & \downarrow \\ L'_{M,N} & \longrightarrow & L'_M \end{array}$$

where  $L'_M$  and  $L'_{M,N}$  are the quotients of  $L_M$  and  $L_{M,N}$  by the subalgebra  $L_M^0$  of vector fields on  $M$  whose infinite jet vanish at points of  $N$ .

The corresponding geometric diagramm is

$$\begin{array}{ccc} \Gamma_{M,N} & \longrightarrow & \Gamma_M \\ \downarrow & & \downarrow \pi \\ \Gamma'_{M,N} & \longrightarrow & \Gamma'_M \end{array}$$

where  $\Gamma'_M$  denotes the space of sections of  $E$  restricted to  $N$  and  $\Gamma'_{M,N}$  the space of sections of  $E'$ . The vertical maps associate to a section its restriction above  $N$ .

$\pi$  is a fibration and  $\Gamma_{M,N}$  is the fiber product of  $\Gamma_M$  and  $\Gamma'_{M,N}$  over  $\Gamma'_M$ .

The spectral sequence of the fibration  $\Gamma_{M,N} \rightarrow \Gamma'_{M,N}$  will correspond to the Hochschild-Serre spectral sequence [16] associated to the ideal  $L_M^0$  in  $L_{M,N}$  (using continuous cochains). The DG-algebra  $C^*(L_M^0)$  will be a model for the fiber.

We assume that we can represent the inclusion of  $N$  in  $M$  by a surjection  $r: A \rightarrow B$  of DG-algebras which are finite dimensional and such that  $A^i = 0$  for  $i > n = \dim M$  and  $B^i = 0$  for  $i > p = \dim N$ .

This is possible in particular if  $M$  and  $N$  are simply connected with finite dimensional real cohomology.

Let  $a_1, \dots, a_s, b_1, \dots, b_t$  be a basis of  $A$  such that the  $a'_i$ 's form a basis of the kernel  $\bar{A}$  of  $r$ . Hence the  $r(b_j)$ 's form a basis of  $B$ .

Let  $\Lambda(x_\alpha)$  (resp.  $\Lambda(y_\lambda)$ ) be a minimal model for  $F_n$  (resp.  $F_{n,p}$ ), or equivalently of  $WU_n$  (resp.  $WU_{n,p}$ ). Then the bundle  $E$  (resp.  $E'$ ) has a minimal model of the form  $A \otimes \Lambda(x_\alpha)$  (resp.  $B \otimes \Lambda(y_\lambda)$ ), where the differential is twisted by terms depending on the choice of representatives for the Pontrjagin classes of  $M$  (cf. [13]).

A model for  $\Gamma_{M,N}$  will be the free algebra  $\Lambda(x_\alpha^i, y_\lambda^j)$  on generators  $x_\alpha^i$ ,  $i = 1, \dots, s$ , and  $y_\lambda^j$ ,  $j = 1, \dots, t$ ,  $\deg x_\alpha^i = \deg x - \deg a_i$ ,  $\deg y_\lambda^j = \deg y_\lambda - \deg b_j$ .

To get the differential, we proceed as follows. Recall that a model for  $\Gamma_M$  is the algebra  $\Lambda(x_\alpha^i, z_\alpha^j)$ ,  $\deg z_\alpha^j = \deg x_\alpha - \deg b_j$ , with a suitable differential (cf. [18], [13] or § 5 with  $G$  the identity). Also models for  $\Gamma'_{M,N}$  and  $\Gamma'_M$  are of the form  $\Lambda(y_\lambda^j)$  and  $\Lambda(z_\alpha^i)$ , resp. with suitable differentials. One has DG-algebra maps

$$\begin{aligned} \Lambda(z_\alpha^j) &\rightarrow \Lambda(x_\alpha^j, z_\alpha^j) \\ \Lambda(z_\alpha^j) &\rightarrow \Lambda(y_\lambda^j) \end{aligned}$$

which are models for the maps  $\Gamma_M \rightarrow \Gamma'_M$  and  $\Gamma'_{M,N} \rightarrow \Gamma'_M$ . The first one is obvious and the second one is completely characterized by the map  $WU_n \rightarrow WU_{n,p}$ .

Now we get the differential on  $A(x^j_\alpha, y^j_\lambda)$  by considering this algebra as the tensor product over  $A(z^j_\alpha)$  of  $A(x^j_\alpha, z^j_\lambda)$  with  $A(y^j_\lambda)$ .

One can make a similar construction using for  $A$  and  $B$  the  $DG$ -algebras  $\Omega_M$  and  $\Omega_N$  of differential forms on  $M$  and  $N$ . Of course one has to work again in more intrinsic terms and use the  $C^\infty$ -topology on  $\Omega_M$  and  $\Omega_N$  (compare with [13]). In this way one gets a  $DG$ -algebra which is also a model for  $\Gamma_{M,N}$  (in fact one proves directly that it is a model for the  $DG$ -algebra constructed above), with a map in  $C^*(L_{M,N})$  inducing an isomorphism in cohomology.

Summing up, we get the following result.

**THEOREM.** *Assume that the inclusion of  $N$  in  $M$  has a model which is a surjection of finite dimensional  $DG$ -algebras. One can construct explicitly a model for  $C^*(L_{M,N})$  which is finite dimensional in each degree.*

*Example.* Suppose that  $M$  is the disk  $D^2$  and  $N$  its boundary  $\partial D^2 = S^1$ . As the inclusion of  $F_{2,1}$  in  $F_2$  is homotopically trivial (equivalently the morphism  $WU_2 \rightarrow WU_{2,1}$  is homotopic to zero), the bundle  $\Gamma_{M,N} \rightarrow \Gamma'_{M,N}$  is trivial.  $WU_2$  is a model for  $S^5 \vee S^5 \vee S^7 \vee S^8 \vee S^8$  and  $WU_{2,1}$  for  $S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$ .

Hence  $C^*(L_D^2, {}_{\partial D^2})$  is a model for the space which is the product of the space of maps of  $S^1$  in  $S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$  with the second loop space of  $S^5 \vee S^5 \vee S^7 \vee S^8 \vee S^8$ .

One can write down quite explicitly the minimal model for that space, but it is harder to compute the cohomology of the first factor. It has an infinite number of multiplicative generators.

## 10. SOME OTHER PROBLEMS

1. As coefficient for the Gelfand-Fuks cochains, one might consider, instead of the field  $R$  with the trivial action of  $L_M$ , a topological  $L_M$ -algebra  $A$ . The problem is to find a model for the  $DG$ -algebra  $C^*(L_M, A)$  of continuous multilinear alternate forms on  $L_M$  with values in  $A$ . The differential is defined by the usual formula involving the action of  $L_M$  on  $A$ .