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Similarly, one can prove that $S_B^*(A \otimes V, B)$ is effectively a model for the space of sections Γ_G (cf. [14]).

Eventually for computations, one proves that one gets also a model for Γ_G by using instead of Ω_{M_G} a DG-algebra A as in § 5 which is a finite dimensional free B-module.

7. Example of a computation

Let us consider the case where M is the n-sphere S^n, G the rotation group SO_{n+1} and E the bundle described in § 3. A model for M_G is the DG-algebra A defined by

$$A = R[p_1, \ldots, p_k, s] / (s^2 - p_k) \quad d \equiv 0 \quad \text{for} \quad n = 2k$$

$$\text{or} \quad A = R[p_1, \ldots, p_{k-1}, \chi] \otimes E(s) \quad ds = \chi \quad \text{for} \quad n = 2k-1$$

where $\deg p_i = 4i$ and $\deg s = n$.

A model for E_G is obtained by taking the tensor product of A with WU_n, the differential being defined by

$$dh_i = c_i - p_{i/2} \quad \text{and} \quad dc_i = 0 .$$

By the way, WSO_n is also a model for E_G.

We now consider the case $n = 2$. The minimal model of E_G is the DG-algebra which begins as

$$A \otimes \Lambda(x_1, x_2, x_3, x_4, x_5, x_{12}, x_{13}, x_{23}, \ldots)$$

where

$$\deg x_1 = \deg x_2 = 5, \deg x_3 = 7, \deg x_4 = \deg x_5 = 8 ,$$
$$\deg x_{12} = 9, \deg x_{13} = \deg x_{23} = 11 ,$$

etc.

(there is an infinite number of generators).

The differential is defined by

$$dx_1 = dx_2 = 0, dx_3 = -p_1^2, dx_4 = p_1 x_1, dx_5 = p_1 x_2 ,$$
$$dx_{12} = x_1 x_2, dx_{13} = x_1 x_3 - p_1 x_4, dx_{23} = x_2 x_3 - p_1 x_5 ,$$

etc.

According to the construction of § 5, a minimal model for the bundle $\Gamma_G \to B_G$ begins as

$$R[p_1] \otimes \Lambda(\bar{x}_1, \bar{x}_2, x_1, x_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, x_3, \bar{x}_{12}, x_4, x_5, \ldots)$$

where

$$\deg \bar{x}_1 = \deg x_i - 2, \varepsilon(x_i) = 1 \otimes x_i + s \otimes \bar{x}_i,$$

dx_i is as above and $dx_{12} = x_1 x_2 + p_1 \bar{x}_1 \bar{x}_2$

$$d\bar{x}_1 = d\bar{x}_2 = d\bar{x}_3 = 0, \; d\bar{x}_4 = p_1 \bar{x}_1, d\bar{x}_5 = p_1 \bar{x}_2,$$
$$d\bar{x}_{12} = \bar{x}_1 x_2 + x_1 \bar{x}_2,$$

etc.

A basis for $H^*(\Gamma_G) = H^*(L_{S^2}, SO_3)$ is given by the classes of the cocycles

$$\bar{x}_1, \bar{x}_2, p_1, x_1, x_2, \bar{x}_3, \bar{x}_1\bar{x}_2, x_1\bar{x}_1, x_1\bar{x}_2, x_2\bar{x}_2, \bar{x}_1\bar{x}_3,$$
$$\bar{x}_2\bar{x}_3, \bar{x}_1\bar{x}_4, \bar{x}_2\bar{x}_5, \bar{x}_1\bar{x}_5 + \bar{x}_2\bar{x}_4, p_1\bar{x}_3\,,$$

etc.

The first multiplicative relations are

$$p_1\bar{x}_1 \sim 0, p_1\bar{x}_2 \sim 0, \bar{x}_1 x_2 \sim \bar{x}_2 x_1, p_1^2 \sim 0, \text{ etc.}$$

The first "exotic" class is given by the cocycle $\bar{x}_1\bar{x}_2\bar{x}_{12}$ of degree 13.

The classes $\bar{x}_1$ and $\bar{x}_2$ correspond to the classes described by Raoul in his lecture [4], for $n = 2$.

We now give an example of a general statement

THEOREM. *The kernel of the map*

$$H^*(BSO_{n+1}) \to H^*(L_{S^n}, SO_{n+1})$$

is the ideal generated by the elements which are polynomials of *degree* $> 2n$ *in the Pontrjagin classes* $p_1, ..., p_{[n/2]}$.

As a consequence, we get exactly what is implied by the vanishing theorem of Bott [1]. For instance, for n odd, the image of the powers of the Euler class is non zero. So one can ask for examples of flat $(2k+1)$-sphere bundles with a non zero power of the Euler class.

One can also check that the homomorphism (see end of § 3)

$$WSO_n \to C^*(L_{S^n}, SO_{n+1}, \Omega_{S^n})$$

induces an injection in cohomology.