

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	24 (1978)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
 Artikel:	ON THE GELFAND-FUKS COHOMOLOGY
Autor:	Haefliger, André
Kapitel:	5. Construction of an algebraic model for the space OF SECTIONS OF A FIBER BUNDLE ([20], [18], [13]).
DOI:	https://doi.org/10.5169/seals-49696

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 23.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

a morphism of A in $C^*(L_M)$. Then one proves directly that it induces an isomorphism in cohomology. The fact that A is also a model for Γ was proved in a similar way (cf. [14]).

When M has a finite dimensional model, one can construct a model for Γ which is finite dimensional in each degree, and with it one can make explicit calculations.

Note that the inclusion $C_{\Delta}^*(L_M, \Omega_M) \rightarrow C^*(L_M, \Omega_M)$ is a model for the evaluation map $\Gamma \times M \rightarrow E$ associating to a section s and a point x of M the element $s(x)$ of E .

For computations along the lines of the spectral sequence of Gelfand-Fuks, see Cohen and Taylor [22].

The proof of theorem 1' is very similar to the proof of theorem 1. In the next paragraph, we explain the construction of an algebraic model for Γ_G suitable for computations. In § 6, we indicate briefly why this is a model for Γ_G .

5. CONSTRUCTION OF AN ALGEBRAIC MODEL FOR THE SPACE OF SECTIONS OF A FIBER BUNDLE ([20], [18], [13]).

As a guide, consider first the geometric situation. Let $p: E \rightarrow M$ be a fiber bundle with base space M , fiber F and let Γ be the space of continuous sections of E .

We have the commutative diagramm

1)

$$\begin{array}{ccc}
 & e & \\
 M \times \Gamma & \xrightarrow{\quad} & E \\
 \downarrow & \searrow & \downarrow p \\
 \Gamma & M & \\
 & \searrow & \downarrow \\
 & & *
 \end{array}$$

where e is the evaluation map associating to the point x of M and the section s the point $s(x)$ of E . The other maps are natural projections (* is a point).

Suppose that a topological group G acts on M and also on E in a way compatible with p . Then G acts also on Γ , and all the maps in the diagramm are equivariant.

For a space X on which G acts, let us denote by X_G the bundle with fiber X associated to the principal universal G -bundle P with base space BG ($= {}^*G$).

From 1) we get the corresponding commutative diagramm

2)

$$\begin{array}{ccc}
 (M \times \Gamma)_G & \longrightarrow & E_G \\
 \downarrow & \searrow & \swarrow \\
 \Gamma_G & & M_G \\
 & \searrow & \downarrow \\
 & & BG
 \end{array}$$

We try now to construct an algebraic analogue of this diagramm. *We assume that the connectivity of the fiber F of E is bigger than the dimension n of M .*

Choose a DG -algebra B which is a model of BG and assume that we can represent the bundle M_G by a DG -algebra A , the projection being represented by a morphism $B \rightarrow A$, and such that A , as a module over B , is free and finite dimensional with a basis s_1, \dots, s_k , where the degree of s_i is not bigger than n (see examples below).

Then we construct the Postnikov decomposition (or minimal model) of the bundle $E_G \rightarrow M_G$. Algebraically, this means that we take a model for E_G which is a tensor product $A \otimes \Lambda(x_\alpha)$, where $\Lambda(x_\alpha)$ is a free graded algebra on an ordered set of generators x_α , the differential of each x_α , being in the subalgebra generated by A and the preceding x_β . Of course the natural inclusion of A in $A \otimes \Lambda(x_\alpha)$ has to be a model for the projection $E_G \rightarrow M_G$. Such a model, with a finite number of generators x_α in each degree, always exists if F is 1-connected and with finite dimensional cohomology, and if G is a connected Lie group (cf. [13], [18]).

A model for Γ_G will be the algebra $B \otimes \Lambda(x_\alpha^i)$, where $\Lambda(x_\alpha^i)$ is the free algebra on generators x_α^i , $i = 1, \dots, k$, and $\deg x_\alpha^i = \deg x_\alpha - \deg s^i$. By our assumptions, $\deg x_\alpha^i > 0$.

A model for the map e will be the morphism

$$\varepsilon : A \otimes \Lambda(x_\alpha) \rightarrow A \otimes \Lambda(x_\alpha^i)$$

of A -algebras defined by

$$\varepsilon(1 \otimes x_\alpha) = \sum_i s^i \otimes x_\alpha^i.$$

The differential on $B \otimes \Lambda(x^i)$ is then uniquely defined by the conditions that $B \otimes \Lambda(x^i)$ should be a DG -algebra over B and that ε should commute with the differential given by the isomorphism with $A \otimes_B (B \otimes \Lambda(x_\alpha^i))$.

The algebraic analogue of diagramm 2) is the commutative diagramm of DG -algebras

2)

$$\begin{array}{ccc}
 A \otimes_B (B \otimes \Lambda(x_\alpha^i)) & \xleftarrow{\quad} & A \otimes \Lambda(x_\alpha) \\
 \uparrow & \swarrow & \nearrow \\
 B \otimes \Lambda(x_\alpha^i) & & A \\
 \uparrow & \swarrow & \uparrow \\
 & & B
 \end{array}$$

Examples.

1. For M , take the 2-sphere S^2 and for E the trivial bundle $S^2 \times S^4$, so that Γ is the space of continuous maps of S^2 in S^4 . The group G will be the rotation group SO_3 acting on S^2 as usual and trivially on S^4 .

As model B for BG we take the polynomial algebra $R[p_1]$ in a generator p_1 of degree 4. A model for M_G is the algebra A quotient of the polynomial algebra $\Lambda(s, p_1)$, where $\deg s = 2$, by the ideal generated by $s^2 - p_1$. The differential is zero. The elements 1 and s form a basis for the B -module A .

A minimal model for the bundle E_G is $A \otimes \Lambda(x, y)$, where $\Lambda(x, y)$ is the free algebra with generators x of degree 4, and y of degree 7, and $dy = x^2$.

According to the preceding recipe, a model for Γ_G is the algebra $R[p_1] \otimes \Lambda(x, y, \bar{x}, \bar{y})$ with $\deg \bar{x} = 2$, $\deg \bar{y} = 5$, the image of x by ε being $1 \otimes x + s \otimes \bar{x}$, similarly for y . The differential is given by $dx = d\bar{x} = 0$, $dy = x^2 + p_1 \bar{x}^2$, $d\bar{y} = 2x\bar{x}$.

2. Take M as the circle, E as the product $S^1 \times F$, where F is a simply connected space, so that Γ is just the space of continuous maps of S^1 in F (case studied by Sullivan [19]). For G we take the group of rotations of the circle, acting trivially on F .

Represent F by its minimal model $\Lambda(x_\alpha)$. A model B for BG is the polynomial algebra $R[e]$ in a generator e of degree 2 and a model A for M_G

is the free commutative algebra $\Lambda(s, e)$, where $\deg s = 1$ and $ds = e$. As a B -module, it is free with basis 1 and s . A model for E_G is just $A \otimes \Lambda(x_\alpha)$.

As model for Γ_G , we take $R[e] \otimes \Lambda(x_\alpha, \bar{x}_\alpha)$, where $\deg \bar{x}_\alpha = \deg x_\alpha - 1$, the image of x_α by ε being $1 \otimes x_\alpha + s \otimes \bar{x}_\alpha$. The differential d is described as follows (compare with Sullivan [18] or [19]). Let h be the derivation of degree -1 of $\Lambda(x_\alpha, \bar{x}_\alpha)$ given by $hx_\alpha = \bar{x}_\alpha$ and $h\bar{x}_\alpha = 0$. Then if d_0 denotes the differential in $\Lambda(x_\alpha)$ identified to a subalgebra of $\Lambda(x_\alpha, \bar{x}_\alpha)$, we have

$$de = 0, dx_\alpha = d_0 x_\alpha - e \bar{x}_\alpha, d\bar{x}_\alpha = -hd_0 x_\alpha$$

Remark. In the case where E is the bundle described in § 4, its minimal model $A \otimes \Lambda(x_\alpha)$ over M_G is complicated, because there is an infinite number of generators x_α (except for $n=1$) labelled by a basis of the rational homotopy of a wedge of spheres, so by a basis of the free graded Lie algebra $L(n)$ generated by the spheres of this wedge (cf. [13]).

6. SKETCH OF THE PROOF OF THE MAIN THEOREM AND APPLICATIONS

We represent the universal principal G -bundle as a limit of finite dimensional bundles P_k and we denote by Ω_P the inverse limit of algebras of forms Ω_{P_k} .

First note that we can replace $C^*(L_M; G)$ by the DG -algebra $C^*(L_M, \Omega_P)_G$ of G -basic elements in $C^*(L_M, \Omega_P)$ (compare with Cartan [5], exposé 20).

A model for E_G will be the algebra $C_\Delta^*(L_M, \Omega_{M \times P})_G = [C_\Delta^*(L_M, \Omega_M \hat{\otimes} \Omega_P)]_G$ and a model for the evaluation map will be the inclusion of this DG -algebra in $C^*(L_M, \Omega_{M \times P})_G$.

In the construction of § 5, we choose $B = \Omega_{BG}$ as model for BG and, instead of taking for A a finite dimensional module over B , we take the DG -algebra $\Omega_{M_G} \approx [\Omega_{M \times P}]_G$ as model for M_G . We have to build the model for Γ_G along the same lines as in § 5, but in more intrinsic terms like in [13]. The minimal model (or Postnikov decomposition of E_G) will be of the form $A \otimes S^*(V)$, where $S^*(V)$ denotes the algebra of symmetric multilinear forms on a graded vector space V (cf. [13]).

As an algebra, the model for Γ_G will be the algebra $S_B^*(A \otimes V, B)$ of continuous symmetric B -multilinear forms on the graded B -module $A \otimes V$. One can construct a map of this model in $C^*(L_M, \Omega_{M \times P})_G$ and prove that it induces an isomorphism in cohomology.