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a morphism of  $A$  in  $C^*(L_M)$ . Then one proves directly that it induces an isomorphism in cohomology. The fact that  $A$  is also a model for  $\Gamma$  was proved in a similar way (cf. [14]).

When  $M$  has a finite dimensional model, one can construct a model for  $\Gamma$  which is finite dimensional in each degree, and with it one can make explicit calculations.

Note that the inclusion  $C_{\Delta}^*(L_M, \Omega_M) \rightarrow C^*(L_M, \Omega_M)$  is a model for the evaluation map  $\Gamma \times M \rightarrow E$  associating to a section  $s$  and a point  $x$  of  $M$  the element  $s(x)$  of  $E$ .

For computations along the lines of the spectral sequence of Gelfand-Fuks, see Cohen and Taylor [22].

The proof of theorem 1' is very similar to the proof of theorem 1. In the next paragraph, we explain the construction of an algebraic model for  $\Gamma_G$  suitable for computations. In § 6, we indicate briefly why this is a model for  $\Gamma_G$ .

## 5. CONSTRUCTION OF AN ALGEBRAIC MODEL FOR THE SPACE OF SECTIONS OF A FIBER BUNDLE ([20], [18], [13]).

As a guide, consider first the geometric situation. Let  $p: E \rightarrow M$  be a fiber bundle with base space  $M$ , fiber  $F$  and let  $\Gamma$  be the space of continuous sections of  $E$ .

We have the commutative diagramm

$$\begin{array}{ccc}
 & & e \\
 & & \longrightarrow \\
 M \times \Gamma & \xrightarrow{\quad} & E \\
 \downarrow & \searrow & \swarrow p \\
 & & M \\
 \downarrow & & \downarrow \\
 \Gamma & & * \\
 & \searrow & \\
 & & *
 \end{array}$$

1)

where  $e$  is the evaluation map associating to the point  $x$  of  $M$  and the section  $s$  the point  $s(x)$  of  $E$ . The other maps are natural projections (\* is a point).

Suppose that a topological group  $G$  acts on  $M$  and also on  $E$  in a way compatible with  $p$ . Then  $G$  acts also on  $\Gamma$ , and all the maps in the diagramm are equivariant.

For a space  $X$  on which  $G$  acts, let us denote by  $X_G$  the bundle with fiber  $X$  associated to the principal universal  $G$ -bundle  $P$  with base space  $BG (= *_G)$ .

From 1) we get the corresponding commutative diagramm

$$\begin{array}{ccc}
 (M \times \Gamma)_G & \longrightarrow & E_G \\
 \downarrow & \searrow & \swarrow \\
 \Gamma_G & & M_G \\
 & \searrow & \downarrow \\
 & & BG
 \end{array}$$

2)

We try now to construct an algebraic analogue of this diagramm. *We assume that the connectivity of the fiber  $F$  of  $E$  is bigger than the dimension  $n$  of  $M$ .*

Choose a  $DG$ -algebra  $B$  which is a model of  $BG$  and assume that we can represent the bundle  $M_G$  by a  $DG$ -algebra  $A$ , the projection being represented by a morphism  $B \rightarrow A$ , and such that  $A$ , as a module over  $B$ , is free and finite dimensional with a basis  $s_1, \dots, s_k$ , where the degree of  $s_i$  is not bigger than  $n$  (see examples below).

Then we construct the Postnikov decomposition (or minimal model) of the bundle  $E_G \rightarrow M_G$ . Algebraically, this means that we take a model for  $E_G$  which is a tensor product  $A \otimes \Lambda(x_\alpha)$ , where  $\Lambda(x_\alpha)$  is a free graded algebra on an ordered set of generators  $x_\alpha$ , the differential of each  $x_\alpha$ , being in the subalgebra generated by  $A$  and the preceding  $x_\beta$ . Of course the natural inclusion of  $A$  in  $A \otimes \Lambda(x_\alpha)$  has to be a model for the projection  $E_G \rightarrow M_G$ . Such a model, with a finite number of generators  $x_\alpha$  in each degree, always exists if  $F$  is 1-connected and with finite dimensional cohomology, and if  $G$  is a connected Lie group (cf. [13], [18]).

A model for  $\Gamma_G$  will be the algebra  $B \otimes \Lambda(x^i_\alpha)$ , where  $\Lambda(x^i_\alpha)$  is the free algebra on generators  $x^i_\alpha$ ,  $i = 1, \dots, k$ , and  $\deg x^i_\alpha = \deg x_\alpha - \deg s^i$ . By our assumptions,  $\deg x^i_\alpha > 0$ .

A model for the map  $e$  will be the morphism

$$\varepsilon: A \otimes \Lambda(x_\alpha) \rightarrow B \otimes \Lambda(x^i_\alpha)$$

of  $A$ -algebras defined by

$$\varepsilon(1 \otimes x_\alpha) = \sum_i s^i \otimes x_\alpha^i.$$

The differential on  $B \otimes \Lambda(x^i)$  is then uniquely defined by the conditions that  $B \otimes \Lambda(x^i)$  should be a  $DG$ -algebra over  $B$  and that  $\varepsilon$  should commute with the differential given by the isomorphism with  $A \otimes_B (B \otimes \Lambda(x^i_\alpha))$ .

The algebraic analogue of diagramm 2) is the commutative diagramm of  $DG$ -algebras

$$\begin{array}{ccccc}
 & & A \otimes_B (B \otimes \Lambda(x^i_\alpha)) & \longleftarrow & A \otimes \Lambda(x_\alpha) \\
 & & \uparrow & \swarrow & \nearrow \\
 2) & & B \otimes \Lambda(x^i_\alpha) & & A \\
 & & \swarrow & \uparrow & \\
 & & & B & 
 \end{array}$$

*Examples.*

1. For  $M$ , take the 2-sphere  $S^2$  and for  $E$  the trivial bundle  $S^2 \times S^4$ , so that  $\Gamma$  is the space of continuous maps of  $S^2$  in  $S^4$ . The group  $G$  will be the rotation group  $SO_3$  acting on  $S^2$  as usual and trivially on  $S^4$ .

As model  $B$  for  $BG$  we take the polynomial algebra  $R[p_1]$  in a generator  $p_1$  of degree 4. A model for  $M_G$  is the algebra  $A$  quotient of the polynomial algebra  $\Lambda(s, p_1)$ , where  $\deg s = 2$ , by the ideal generated by  $s^2 - p_1$ . The differential is zero. The elements 1 and  $s$  form a basis for the  $B$ -module  $A$ .

A minimal model for the bundle  $E_G$  is  $A \otimes \Lambda(x, y)$ , where  $\Lambda(x, y)$  is the free algebra with generators  $x$  of degree 4, and  $y$  of degree 7, and  $dy = x^2$ .

According to the preceding recipe, a model for  $\Gamma_G$  is the algebra  $R[p_1] \otimes \Lambda(x, y, \bar{x}, \bar{y})$  with  $\deg \bar{x} = 2$ ,  $\deg \bar{y} = 5$ , the image of  $x$  by  $\varepsilon$  being  $1 \otimes x + s \otimes \bar{x}$ , similarly for  $y$ . The differential is given by  $dx = d\bar{x} = 0$ ,  $dy = x^2 + p_1 \bar{x}^2$ ,  $d\bar{y} = 2x\bar{x}$ .

2. Take  $M$  as the circle,  $E$  as the product  $S^1 \times F$ , where  $F$  is a simply connected space, so that  $\Gamma$  is just the space of continuous maps of  $S^1$  in  $F$  (case studied by Sullivan [19]). For  $G$  we take the group of rotations of the circle, acting trivially on  $F$ .

Represent  $F$  by its minimal model  $\Lambda(x_\alpha)$ . A model  $B$  for  $BG$  is the polynomial algebra  $R[e]$  in a generator  $e$  of degree 2 and a model  $A$  for  $M_G$

is the free commutative algebra  $\Lambda(s, e)$ , where  $\deg s = 1$  and  $ds = e$ . As a  $B$ -module, it is free with basis 1 and  $s$ . A model for  $E_G$  is just  $\Lambda \otimes \Lambda(x_\alpha)$ .

As model for  $\Gamma_G$ , we take  $R[e] \otimes \Lambda(x_\alpha, \bar{x}_\alpha)$ , where  $\deg \bar{x}_\alpha = \deg x_\alpha - 1$ , the image of  $x_\alpha$  by  $\epsilon$  being  $1 \otimes x_\alpha + s \otimes \bar{x}_\alpha$ . The differential  $d$  is described as follows (compare with Sullivan [18] or [19]). Let  $h$  be the derivation of degree  $-1$  of  $\Lambda(x_\alpha, \bar{x}_\alpha)$  given by  $hx_\alpha = \bar{x}_\alpha$  and  $h\bar{x}_\alpha = 0$ . Then if  $d_0$  denotes the differential in  $\Lambda(x_\alpha)$  identified to a subalgebra of  $\Lambda(x_\alpha, \bar{x}_\alpha)$ , we have

$$de = 0, dx_\alpha = d_0x_\alpha - e\bar{x}_\alpha, d\bar{x}_\alpha = -hd_0x_\alpha$$

*Remark.* In the case where  $E$  is the bundle described in § 4, its minimal model  $\Lambda \otimes \Lambda(x_\alpha)$  over  $M_G$  is complicated, because there is an infinite number of generators  $x_\alpha$  (except for  $n=1$ ) labelled by a basis of the rational homotopy of a wedge of spheres, so by a basis of the free graded Lie algebra  $L(n)$  generated by the spheres of this wedge (cf. [13]).

## 6. SKETCH OF THE PROOF OF THE MAIN THEOREM AND APPLICATIONS

We represent the universal principal  $G$ -bundle as a limit of finite dimensional bundles  $P_k$  and we denote by  $\Omega_P$  the inverse limit of algebras of forms  $\Omega_{P_k}$ .

First note that we can replace  $C^*(L_M; G)$  by the  $DG$ -algebra  $C^*(L_M, \Omega_P)_G$  of  $G$ -basic elements in  $C^*(L_M, \Omega_P)$  (compare with Cartan [5], exposé 20).

A model for  $E_G$  will be the algebra  $C_\Delta^*(L_M, \Omega_{M \times P})_G = [C_\Delta^*(L_M, \Omega_M \hat{\otimes} \Omega_P)_G]$  and a model for the evaluation map will be the inclusion of this  $DG$ -algebra in  $C^*(L_M, \Omega_{M \times P})_G$ .

In the construction of § 5, we choose  $B = \Omega_{BG}$  as model for  $BG$  and, instead of taking for  $A$  a finite dimensional module over  $B$ , we take the  $DG$ -algebra  $\Omega_{M_G} \approx [\Omega_{M \times P}]_G$  as model for  $M_G$ . We have to build the model for  $\Gamma_G$  along the same lines as in § 5, but in more intrinsic terms like in [13]. The minimal model (or Postnikov decomposition of  $E_G$ ) will be of the form  $A \otimes S^*(V)$ , where  $S^*(V)$  denotes the algebra of symmetric multilinear forms on a graded vector space  $V$  (cf. [13]).

As an algebra, the model for  $\Gamma_G$  will be the algebra  $S_B^*(A \otimes V, B)$  of continuous symmetric  $B$ -multilinear forms on the graded  $B$ -module  $A \otimes V$ . One can construct a map of this model in  $C^*(L_M, \Omega_{M \times P})_G$  and prove that it induces an isomorphism in cohomology.