Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

**Band:** 24 (1978)

**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE GELFAND-FUKS COHOMOLOGY

Autor: Haefliger, André

**Kapitel:** 5. Construction of an algebraic model for the space OF SECTIONS OF

A FIBER BUNDLE ([20], [18], [13]).

**DOI:** https://doi.org/10.5169/seals-49696

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a morphism of A in  $C^*(L_M)$ . Then one proves directly that it induces an isomorphism in cohomology. The fact that A is also a model for  $\Gamma$  was proved in a similar way (cf. [14]).

When M has a finite dimensional model, one can construct a model for  $\Gamma$  which is finite dimensional in each degree, and with it one can make explicit calculations.

Note that the inclusion  $C_{\triangle}^*(L_M, \Omega_M) \to C^*(L_M, \Omega_M)$  is a model for the evaluation map  $\Gamma \times M \to E$  associating to a section s and a point x of M the element s(x) of E.

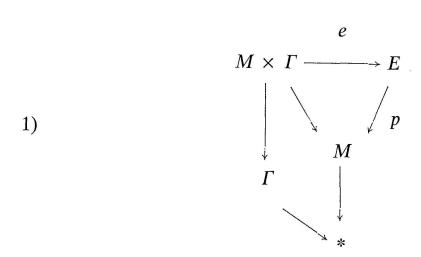
For computations along the lines of the spectral sequence of Gelfand-Fuks, see Cohen and Taylor [22].

The proof of theorem 1' is very similar to the proof of theorem 1. In the next paragraph, we explain the construction of an algebraic model for  $\Gamma_G$  suitable for computations. In § 6, we indicate briefly why this is a model for  $\Gamma_G$ .

# 5. Construction of an algebraic model for the space of sections of a fiber bundle ([20], [18], [13]).

As a guide, consider first the geometric situation. Let  $p: E \to M$  be a fiber bundle with base space M, fiber F and let  $\Gamma$  be the space of continuous sections of E.

We have the commutative diagramm

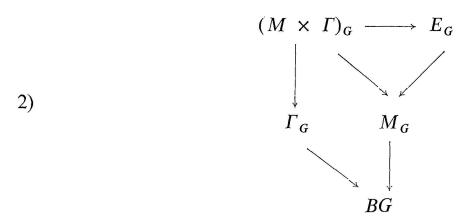


where e is the evaluation map associating to the point x of M and the section s the point s(x) of E. The other maps are natural projections (\* is a point).

Suppose that a topological group G acts on M and also on E in a way compatible with p. Then G acts also on  $\Gamma$ , and all the maps in the diagramm are equivariant.

For a space X on which G acts, let us denote by  $X_G$  the bundle with fiber X associated to the principal universal G-bundle P with base space  $BG = *_G$ .

From 1) we get the corresponding commutative diagramm



We try now to construct an algebraic analogue of this diagramm. We assume that the connectivity of the fiber F of E is bigger than the dimension n of M.

Choose a DG-algebra B which is a model of BG and assume that we can represent the bundle  $M_G$  by a DG-algebra A, the projection being represented by a morphism  $B \to A$ , and such that A, as a module over B, is free and finite dimensional with a basis  $s_1, ..., s_k$ , where the degree of  $s_i$  is not bigger than n (see examples below).

Then we construct the Postnikov decomposition (or minimal model) of the bundle  $E_G o M_G$ . Algebraically, this means that we take a model for  $E_G$  which is a tensor product  $A \otimes A(x_\alpha)$ , where  $A(x_\alpha)$  is a free graded algebra on an ordered set of generators  $x_\alpha$ , the differential of each  $x_\alpha$ , being in the subalgebra generated by A and the preceding  $x_\beta$ . Of course the natural inclusion of A in  $A \otimes A(x_\alpha)$  has to be a model for the projection  $E_G \to M_G$ . Such a model, with a finite number of generators  $x_\alpha$  in each degree, always exists if F is 1-connected and with finite dimensional cohomology, and if G is a connected Lie group (cf. [13], [18]).

A model for  $\Gamma_G$  will be the algebra  $B \otimes \Lambda(x^i_{\alpha})$ , where  $\Lambda(x^i_{\alpha})$  is the free algebra on generators  $x^i_{\alpha}$ , i=1,...,k, and  $\deg x^i_{\alpha}=\deg x_{\alpha}-\deg s^i$ . By our assumptions,  $\deg x^i_{\alpha}>0$ .

A model for the map e will be the morphism

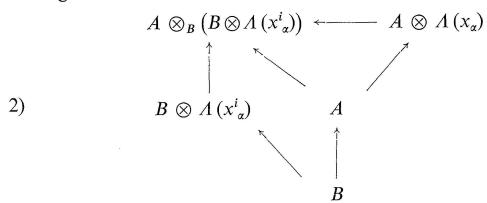
$$\varepsilon: A \otimes \Lambda(x_{\alpha}) \to A \otimes \Lambda(x_{\alpha}^{i})$$

of A-algebras defined by

$$\varepsilon(1\otimes x_{\alpha}) = \sum_{i} s^{i} \otimes x_{\alpha}^{i}.$$

The differential on  $B \otimes \Lambda(x^i)$  is then uniquely defined by the conditions that  $B \otimes \Lambda(x^i)$  should be a DG-algebra over B and that  $\varepsilon$  should commute with the differential given by the isomorphism with  $A \otimes_B (B \otimes \Lambda(x^i_{\alpha}))$ .

The algebraic analogue of diagramm 2) is the commutative diagramm of DG-algebras



Examples.

1. For M, take the 2-sphere  $S^2$  and for E the trivial bundle  $S^2 \times S^4$ , so that  $\Gamma$  is the space of continuous maps of  $S^2$  in  $S^4$ . The group G will be the rotation group  $SO_3$  acting on  $S^2$  as usual and trivialy on  $S^4$ .

As model B for BG we take the polynomial algebra  $R[p_1]$  in a generator  $p_1$  of degree 4. A model for  $M_G$  is the algebra A quotient of the polynomial algebra  $\Lambda(s, p_1)$ , where deg s = 2, by the ideal generated by  $s^2 - p_1$ . The differential is zero. The elements 1 and s form a basis for the B-module A.

A minimal model for the bundle  $E_G$  is  $A \otimes A(x, y)$ , where A(x, y) is the free algebra with generators x of degree 4, and y of degree 7, and  $dy = x^2$ .

According to the preceding recipe, a model for  $\Gamma_G$  is the algebra  $R[p_1] \otimes \Lambda(x, y, \bar{x}, \bar{y})$  with  $\deg \bar{x} = 2$ ,  $\deg \bar{y} = 5$ , the image of x by  $\varepsilon$  being  $1 \otimes x + s \otimes \bar{x}$ , similarly for y. The differential is given by  $dx = d\bar{x} = 0$ ,  $dy = x^2 + p_1\bar{x}^2$ ,  $d\bar{y} = 2x\bar{x}$ .

2. Take M as the circle, E as the product  $S^1 \times F$ , where F is a simply connected space, so that  $\Gamma$  is just the space of continuous maps of  $S^1$  in F (case studied by Sullivan [19]). For G we take the group of rotations of the circle, acting trivially on F.

Represent F by its minimal model  $A(x_{\alpha})$ . A model B for BG is the polynomial algebra R[e] in a generator e of degree 2 and a model A for  $M_G$ 

is the free commutative algebra  $\Lambda(s, e)$ , where deg s = 1 and ds = e. As a *B*-module, it is free with basis 1 and s. A model for  $E_G$  is just  $A \otimes \Lambda(x_\alpha)$ .

As model for  $\Gamma_G$ , we take  $R[e] \otimes \Lambda(x_{\alpha}, \bar{x}_{\alpha})$ , where  $\deg \bar{x}_{\alpha} = \deg x_{\alpha} - 1$ , the image of  $x_{\alpha}$  by  $\varepsilon$  being  $1 \otimes x_{\alpha} + s \otimes \bar{x}_{\alpha}$ . The differential d is described as follows (compare with Sullivan [18] or [19]). Let h be the derivation of degree -1 of  $\Lambda(x_{\alpha}, \bar{x}_{\alpha})$  given by  $hx_{\alpha} = \bar{x}_{\alpha}$  and  $h\bar{x}_{\alpha} = 0$ . Then if  $d_0$  denotes the differential in  $\Lambda(x_{\alpha})$  identified to a subalgebra of  $\Lambda(x_{\alpha}, \bar{x}_{\alpha})$ , we have

$$de = 0, dx_{\alpha} = d_0 x_{\alpha} - e \bar{x}_{\alpha}, d\bar{x}_{\alpha} = -h d_0 x_{\alpha}$$

Remark. In the case where E is the bundle described in § 4, its minimal model  $A \otimes \Lambda(x_{\alpha})$  over  $M_G$  is complicated, because there is an infinite number of generators  $x_{\alpha}$  (except for n=1) labelled by a basis of the rational homotopy of a wedge of spheres, so by a basis of the free graded Lie algebra L(n) generated by the spheres of this wedge (cf. [13]).

## 6. Sketch of the proof of the main theorem and applications

We represent the universal principal G-bundle as a limit of finite dimensional bundles  $P_k$  and we denote by  $\Omega_P$  the inverse limit of algebras of forms  $\Omega_{P_k}$ .

First note that we can replace  $C^*(L_M; G)$  by the DG-algebra  $C^*(L_M, \Omega_P)_G$  of G-basic elements in  $C^*(L_M, \Omega_P)$  (compare with Cartan [5], exposé 20).

A model for  $E_G$  will be the algebra  $C_{\triangle}^*$   $(L_M, \Omega_{M \times P})_G = [C_{\triangle}^* (L_M, \Omega_M \Omega_M \Omega_M)]_G$  and a model for the evaluation map will be the inclusion of this DG-algebra in  $C^* (L_M, \Omega_{M \times P})_G$ .

In the construction of § 5, we choose  $B = \Omega_{BG}$  as model for BG and, instead of taking for A a finite dimensional module over B, we take the DG-algebra  $\Omega_{M_G} \approx [\Omega_{M \times P}]_G$  as model for  $M_G$ . We have to build the model for  $\Gamma_G$  along the same lines as in § 5, but in more intrinsic terms like in [13]. The minimal model (or Postnikov decomposition of  $E_G$ ) will be of the form  $A \otimes S^*(V)$ , where  $S^*(V)$  denotes the algebra of symmetric multilinear forms on a graded vector space V (cf. [13]).

As an algebra, the model for  $\Gamma_G$  will be the algebra  $S_B^*(A \otimes V, B)$  of continuous symmetric *B*-multilinear forms on the graded *B*-module  $A \otimes V$ . One can construct a map of this model in  $C^*(L_M, \Omega_{M \times P})_G$  and prove that it induces an isomorphism in cohomology.