Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 24 (1978)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE GELFAND-FUKS COHOMOLOGY

Autor: Haefliger, André

Kapitel: 5. Construction of an algebraic model for the space OF SECTIONS OF A FIBER BUNDLE ([20], [18], [13]).

DOI: https://doi.org/10.5169/seals-49696

a morphism of A in $C^*(L_M)$. Then one proves directly that it induces an isomorphism in cohomology. The fact that A is also a model for Γ was proved in a similar way (cf. [14]).

When M has a finite dimensional model, one can construct a model for Γ which is finite dimensional in each degree, and with it one can make explicit calculations.

Note that the inclusion $C^*_\Delta(L_M, \Omega_M) \to C^*(L_M, \Omega_M)$ is a model for the evaluation map $\Gamma \times M \to E$ associating to a section s and a point x of M the element $s(x)$ of E.

For computations along the lines of the spectral sequence of Gelfand-Fuks, see Cohen and Taylor [22].

The proof of theorem 1′ is very similar to the proof of theorem 1. In the next paragraph, we explain the construction of an algebraic model for Γ_G suitable for computations. In § 6, we indicate briefly why this is a model for Γ_G.

5. Construction of an algebraic model for the space of sections of a fiber bundle ([20], [18], [13]).

As a guide, consider first the geometric situation. Let $p: E \to M$ be a fiber bundle with base space M, fiber F and let Γ be the space of continuous sections of E.

We have the commutative diagramm

1)
$$\begin{tikzcd} M \times \Gamma \arrow[rr, "e"] \arrow[d] \arrow[dr] & & E \arrow[dl, "p"] \\ \Gamma \arrow[dr] & M \arrow[d] & \\ & * & \end{tikzcd}$$

where e is the evaluation map associating to the point x of M and the section s the point $s(x)$ of E. The other maps are natural projections ($*$ is a point).

Suppose that a topological group G acts on M and also on E in a way compatible with p. Then G acts also on Γ, and all the maps in the diagramm are equivariant.

For a space X on which G acts, let us denote by X_G the bundle with fiber X associated to the principal universal G-bundle P with base space BG $(= *_G)$.

From 1) we get the corresponding commutative diagramm

2)
$$\begin{tikzcd}
(M \times \Gamma)_G \arrow[rr] \arrow[d] \arrow[dr] & & E_G \arrow[dl] \\
\Gamma_G \arrow[dr] & M_G \arrow[d] & \\
& BG &
\end{tikzcd}$$

We try now to construct an algebraic analogue of this diagramm. *We assume that the connectivity of the fiber F of E is bigger than the dimension n of M.*

Choose a DG-algebra B which is a model of BG and assume that *we can represent the bundle M_G by a DG-algebra A, the projection being represented by a morphism $B \to A$, and such that A, as a module over B, is free and finite dimensional with a basis $s_1, \ldots, s_k$, where the degree of s_i is not bigger than n* (see examples below).

Then we construct the Postnikov decomposition (or minimal model) of the bundle $E_G \to M_G$. Algebraically, this means that we take a model for E_G which is a tensor product $A \otimes \Lambda(x_\alpha)$, where $\Lambda(x_\alpha)$ is a free graded algebra on an ordered set of generators x_α, the differential of each x_α, being in the subalgebra generated by A and the preceding x_β. Of course the natural inclusion of A in $A \otimes \Lambda(x_\alpha)$ has to be a model for the projection $E_G \to M_G$. Such a model, with a finite number of generators x_α in each degree, always exists if F is 1-connected and with finite dimensional cohomology, and if G is a connected Lie group (cf. [13], [18]).

A model for Γ_G will be the algebra $B \otimes \Lambda(x^i{}_\alpha)$, where $\Lambda(x^i{}_\alpha)$ is the free algebra on generators $x^i{}_\alpha$, $i = 1, \ldots, k$, and $\deg x^i{}_\alpha = \deg x_\alpha - \deg s^i$. By our assumptions, $\deg x^i{}_\alpha > 0$.

A model for the map e will be the morphism

$$\varepsilon: A \otimes \Lambda(x_\alpha) \to A \otimes \Lambda(x^i{}_\alpha)$$

of A-algebras defined by

$$\varepsilon(1 \otimes x_\alpha) = \sum_i s^i \otimes {x'}^i_\alpha \, .$$

The differential on $B \otimes \Lambda(x^i)$ is then uniquely defined by the conditions that $B \otimes \Lambda(x^i)$ should be a DG-algebra over B and that ε should commute with the differential given by the isomorphism with $A \otimes_B (B \otimes \Lambda({x'}^i_\alpha))$.

The algebraic analogue of diagramm 2) is the commutative diagramm of DG-algebras

$$2) \qquad \begin{tikzcd}
A \otimes_B (B \otimes \Lambda({x'}^i_\alpha)) & & A \otimes \Lambda(x_\alpha) \arrow[ll] \\
B \otimes \Lambda({x'}^i_\alpha) \arrow[u] & A \arrow[ul] \arrow[ur] & \\
 & B \arrow[ul] \arrow[u] &
\end{tikzcd}$$

Examples.

1. For M, take the 2-sphere S^2 and for E the trivial bundle $S^2 \times S^4$, so that Γ is the space of continuous maps of S^2 in S^4. The group G will be the rotation group SO_3 acting on S^2 as usual and trivialy on S^4.

As model B for BG we take the polynomial algebra $R[p_1]$ in a generator p_1 of degree 4. A model for M_G is the algebra A quotient of the polynomial algebra $\Lambda(s, p_1)$, where $\deg s = 2$, by the ideal generated by $s^2 - p_1$. The differential is zero. The elements 1 and s form a basis for the B-module A.

A minimal model for the bundle E_G is $A \otimes \Lambda(x, y)$, where $\Lambda(x, y)$ is the free algebra with generators x of degree 4, and y of degree 7, and $dy = x^2$.

According to the preceding recipe, a model for Γ_G is the algebra $R[p_1] \otimes \Lambda(x, y, \bar{x}, \bar{y})$ with $\deg \bar{x} = 2$, $\deg \bar{y} = 5$, the image of x by ε being $1 \otimes x + s \otimes \bar{x}$, similarly for y. The differential is given by $dx = d\bar{x} = 0$, $dy = x^2 + p_1 \bar{x}^2$, $d\bar{y} = 2x\bar{x}$.

2. Take M as the circle, E as the product $S^1 \times F$, where F is a simply connected space, so that Γ is just the space of continuous maps of S^1 in F (case studied by Sullivan [19]). For G we take the group of rotations of the circle, acting trivially on F.

Represent F by its minimal model $\Lambda(x_\alpha)$. A model B for BG is the polynomial algebra $R[e]$ in a generator e of degree 2 and a model A for M_G

is the free commutative algebra $\Lambda(s, e)$, where $\deg s = 1$ and $ds = e$. As a B-module, it is free with basis 1 and s. A model for E_G is just $A \otimes \Lambda(x_\alpha)$.

As model for Γ_G, we take $R[e] \otimes \Lambda(x_\alpha, \bar{x}_\alpha)$, where $\deg \bar{x}_\alpha = \deg x_\alpha - 1$, the image of x_α by ε being $1 \otimes x_\alpha + s \otimes \bar{x}_\alpha$. The differential d is described as follows (compare with Sullivan [18] or [19]). Let h be the derivation of degree -1 of $\Lambda(x_\alpha, \bar{x}_\alpha)$ given by $hx_\alpha = \bar{x}_\alpha$ and $h\bar{x}_\alpha = 0$. Then if d_0 denotes the differential in $\Lambda(x_\alpha)$ identified to a subalgebra of $\Lambda(x_\alpha, \bar{x}_\alpha)$, we have

$$de = 0, dx_\alpha = d_0 x_\alpha - e\,\bar{x}_\alpha, d\bar{x}_\alpha = -hd_0 x_\alpha$$

Remark. In the case where E is the bundle described in § 4, its minimal model $A \otimes \Lambda(x_\alpha)$ over M_G is complicated, because there is an infinite number of generators x_α (except for $n = 1$) labelled by a basis of the rational homotopy of a wedge of spheres, so by a basis of the free graded Lie algebra $L(n)$ generated by the spheres of this wedge (cf. [13]).

6. Sketch of the proof of the main theorem and applications

We represent the universal principal G-bundle as a limit of finite dimensional bundles P_k and we denote by Ω_P the inverse limit of algebras of forms Ω_{P_k}.

First note that we can replace $C^*(L_M; G)$ by the DG-algebra $C^*(L_M, \Omega_P)_G$ of G-basic elements in $C^*(L_M, \Omega_P)$ (compare with Cartan [5], exposé 20).

A model for E_G will be the algebra $C^*_\Delta(L_M, \Omega_{M \times P})_G = [C^*_\Delta(L_M, \Omega_M \hat{\otimes} \Omega_P]_G$ and a model for the evaluation map will be the inclusion of this DG-algebra in $C^*(L_M, \Omega_{M \times P})_G$.

In the construction of § 5, we choose $B = \Omega_{BG}$ as model for BG and, instead of taking for A a finite dimensional module over B, we take the DG-algebra $\Omega_{M_G} \approx [\Omega_{M \times P}]_G$ as model for M_G. We have to build the model for Γ_G along the same lines as in § 5, but in more intrinsic terms like in [13]. The minimal model (or Postnikov decomposition of E_G) will be of the form $A \otimes S^*(V)$, where $S^*(V)$ denotes the algebra of symmetric multilinear forms on a graded vector space V (cf. [13]).

As an algebra, the model for Γ_G will be the algebra $S^*_B(A \otimes V, B)$ of continuous symmetric B-multilinear forms on the graded B-module $A \otimes V$. One can construct a map of this model in $C^*(L_M, \Omega_{M \times P})_G$ and prove that it induces an isomorphism in cohomology.