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Autor: Haefliger, André

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A FIBER BUNDLE ([20], [18], [13]).

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a morphism of A in $C^*(L_M)$. Then one proves directly that it induces an isomorphism in cohomology. The fact that A is also a model for Γ was proved in a similar way (cf. [14]).

When M has a finite dimensional model, one can construct a model for Γ which is finite dimensional in each degree, and with it one can make explicit calculations.

Note that the inclusion $C_{\triangle}^*(L_M, \Omega_M) \to C^*(L_M, \Omega_M)$ is a model for the evaluation map $\Gamma \times M \to E$ associating to a section s and a point x of M the element s(x) of E.

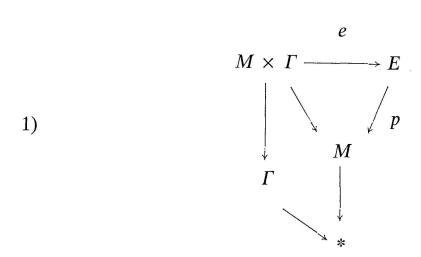
For computations along the lines of the spectral sequence of Gelfand-Fuks, see Cohen and Taylor [22].

The proof of theorem 1' is very similar to the proof of theorem 1. In the next paragraph, we explain the construction of an algebraic model for Γ_G suitable for computations. In § 6, we indicate briefly why this is a model for Γ_G .

5. Construction of an algebraic model for the space of sections of a fiber bundle ([20], [18], [13]).

As a guide, consider first the geometric situation. Let $p: E \to M$ be a fiber bundle with base space M, fiber F and let Γ be the space of continuous sections of E.

We have the commutative diagramm

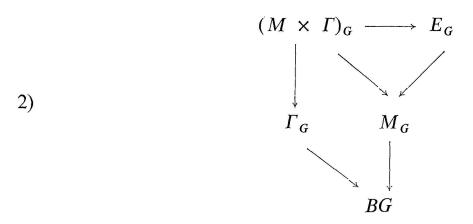


where e is the evaluation map associating to the point x of M and the section s the point s(x) of E. The other maps are natural projections (* is a point).

Suppose that a topological group G acts on M and also on E in a way compatible with p. Then G acts also on Γ , and all the maps in the diagramm are equivariant.

For a space X on which G acts, let us denote by X_G the bundle with fiber X associated to the principal universal G-bundle P with base space $BG = *_G$.

From 1) we get the corresponding commutative diagramm



We try now to construct an algebraic analogue of this diagramm. We assume that the connectivity of the fiber F of E is bigger than the dimension n of M.

Choose a DG-algebra B which is a model of BG and assume that we can represent the bundle M_G by a DG-algebra A, the projection being represented by a morphism $B \to A$, and such that A, as a module over B, is free and finite dimensional with a basis $s_1, ..., s_k$, where the degree of s_i is not bigger than n (see examples below).

Then we construct the Postnikov decomposition (or minimal model) of the bundle $E_G o M_G$. Algebraically, this means that we take a model for E_G which is a tensor product $A \otimes A(x_\alpha)$, where $A(x_\alpha)$ is a free graded algebra on an ordered set of generators x_α , the differential of each x_α , being in the subalgebra generated by A and the preceding x_β . Of course the natural inclusion of A in $A \otimes A(x_\alpha)$ has to be a model for the projection $E_G \to M_G$. Such a model, with a finite number of generators x_α in each degree, always exists if F is 1-connected and with finite dimensional cohomology, and if G is a connected Lie group (cf. [13], [18]).

A model for Γ_G will be the algebra $B \otimes \Lambda(x^i_{\alpha})$, where $\Lambda(x^i_{\alpha})$ is the free algebra on generators x^i_{α} , i=1,...,k, and $\deg x^i_{\alpha}=\deg x_{\alpha}-\deg s^i$. By our assumptions, $\deg x^i_{\alpha}>0$.

A model for the map e will be the morphism

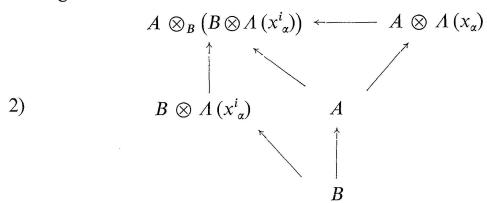
$$\varepsilon: A \otimes \Lambda(x_{\alpha}) \to A \otimes \Lambda(x_{\alpha}^{i})$$

of A-algebras defined by

$$\varepsilon(1\otimes x_{\alpha}) = \sum_{i} s^{i} \otimes x_{\alpha}^{i}.$$

The differential on $B \otimes \Lambda(x^i)$ is then uniquely defined by the conditions that $B \otimes \Lambda(x^i)$ should be a DG-algebra over B and that ε should commute with the differential given by the isomorphism with $A \otimes_B (B \otimes \Lambda(x^i_{\alpha}))$.

The algebraic analogue of diagramm 2) is the commutative diagramm of DG-algebras



Examples.

1. For M, take the 2-sphere S^2 and for E the trivial bundle $S^2 \times S^4$, so that Γ is the space of continuous maps of S^2 in S^4 . The group G will be the rotation group SO_3 acting on S^2 as usual and trivialy on S^4 .

As model B for BG we take the polynomial algebra $R[p_1]$ in a generator p_1 of degree 4. A model for M_G is the algebra A quotient of the polynomial algebra $\Lambda(s, p_1)$, where deg s = 2, by the ideal generated by $s^2 - p_1$. The differential is zero. The elements 1 and s form a basis for the B-module A.

A minimal model for the bundle E_G is $A \otimes A(x, y)$, where A(x, y) is the free algebra with generators x of degree 4, and y of degree 7, and $dy = x^2$.

According to the preceding recipe, a model for Γ_G is the algebra $R[p_1] \otimes \Lambda(x, y, \bar{x}, \bar{y})$ with $\deg \bar{x} = 2$, $\deg \bar{y} = 5$, the image of x by ε being $1 \otimes x + s \otimes \bar{x}$, similarly for y. The differential is given by $dx = d\bar{x} = 0$, $dy = x^2 + p_1\bar{x}^2$, $d\bar{y} = 2x\bar{x}$.

2. Take M as the circle, E as the product $S^1 \times F$, where F is a simply connected space, so that Γ is just the space of continuous maps of S^1 in F (case studied by Sullivan [19]). For G we take the group of rotations of the circle, acting trivially on F.

Represent F by its minimal model $A(x_{\alpha})$. A model B for BG is the polynomial algebra R[e] in a generator e of degree 2 and a model A for M_G

is the free commutative algebra $\Lambda(s, e)$, where deg s = 1 and ds = e. As a *B*-module, it is free with basis 1 and s. A model for E_G is just $A \otimes \Lambda(x_\alpha)$.

As model for Γ_G , we take $R[e] \otimes \Lambda(x_{\alpha}, \bar{x}_{\alpha})$, where $\deg \bar{x}_{\alpha} = \deg x_{\alpha} - 1$, the image of x_{α} by ε being $1 \otimes x_{\alpha} + s \otimes \bar{x}_{\alpha}$. The differential d is described as follows (compare with Sullivan [18] or [19]). Let h be the derivation of degree -1 of $\Lambda(x_{\alpha}, \bar{x}_{\alpha})$ given by $hx_{\alpha} = \bar{x}_{\alpha}$ and $h\bar{x}_{\alpha} = 0$. Then if d_0 denotes the differential in $\Lambda(x_{\alpha})$ identified to a subalgebra of $\Lambda(x_{\alpha}, \bar{x}_{\alpha})$, we have

$$de = 0, dx_{\alpha} = d_0 x_{\alpha} - e \bar{x}_{\alpha}, d\bar{x}_{\alpha} = -h d_0 x_{\alpha}$$

Remark. In the case where E is the bundle described in § 4, its minimal model $A \otimes \Lambda(x_{\alpha})$ over M_G is complicated, because there is an infinite number of generators x_{α} (except for n=1) labelled by a basis of the rational homotopy of a wedge of spheres, so by a basis of the free graded Lie algebra L(n) generated by the spheres of this wedge (cf. [13]).

6. Sketch of the proof of the main theorem and applications

We represent the universal principal G-bundle as a limit of finite dimensional bundles P_k and we denote by Ω_P the inverse limit of algebras of forms Ω_{P_k} .

First note that we can replace $C^*(L_M; G)$ by the DG-algebra $C^*(L_M, \Omega_P)_G$ of G-basic elements in $C^*(L_M, \Omega_P)$ (compare with Cartan [5], exposé 20).

A model for E_G will be the algebra C_{\triangle}^* $(L_M, \Omega_{M \times P})_G = [C_{\triangle}^* (L_M, \Omega_M \Omega_M \Omega_M)]_G$ and a model for the evaluation map will be the inclusion of this DG-algebra in $C^* (L_M, \Omega_{M \times P})_G$.

In the construction of § 5, we choose $B = \Omega_{BG}$ as model for BG and, instead of taking for A a finite dimensional module over B, we take the DG-algebra $\Omega_{M_G} \approx [\Omega_{M \times P}]_G$ as model for M_G . We have to build the model for Γ_G along the same lines as in § 5, but in more intrinsic terms like in [13]. The minimal model (or Postnikov decomposition of E_G) will be of the form $A \otimes S^*(V)$, where $S^*(V)$ denotes the algebra of symmetric multilinear forms on a graded vector space V (cf. [13]).

As an algebra, the model for Γ_G will be the algebra $S_B^*(A \otimes V, B)$ of continuous symmetric *B*-multilinear forms on the graded *B*-module $A \otimes V$. One can construct a map of this model in $C^*(L_M, \Omega_{M \times P})_G$ and prove that it induces an isomorphism in cohomology.