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through a map  $H^*(BG; R) \rightarrow H^*(L_M; G)$  so that we get a commutative diagram

$$\begin{array}{ccc} & & H^*(L_M; G) \\ & \nearrow & \downarrow \\ H^*(BG; R) & & \\ & \searrow & \\ & & H^*(X; R) \end{array}$$

So it is important to compute the map  $H^*(BG; R) \rightarrow H^*(L_M; G)$ . When  $G$  is a compact connected Lie group, then  $H^*(BG; R)$  is the algebra  $I(G)$  of invariant polynomials on the Lie algebra of  $G$ , and the map from  $I(G)$  to  $C^*(L_M; G)$  is given by a  $G$ -connexion in  $C^*(L_M)$  (cf. [5]).

In the example above, namely  $M = S^1$  and  $G = SO_2$ , then  $H^*(BSO_2)$  is a polynomial algebra in a generator of degree 2, the Euler class, which is mapped on a non zero multiple of  $e$ .

### 3. THE FORMAL VECTOR FIELDS AND THE DIAGONAL COMPLEX

Given a point  $x$  on  $M$ , we can consider the Lie algebra  $L_M^x$  of infinite jets at  $x$  of vector fields on  $M$  with the quotient topology. It is isomorphic to the Lie algebra  $\mathfrak{a}_n$  of formal vector fields  $\sum v_i(x) \partial/\partial x^i$  in  $R^n$ , where the  $v_i(x)$  are formal power series in the coordinates  $x^1, \dots, x^n$ .

The natural map  $L_M \rightarrow L_M^x$  associating to a vector field its jet at  $x$  gives a  $DG$ -algebra morphism

$$C^*(L_M^x) \rightarrow C^*(L_M)$$

where  $C^*(L_M^x)$  is the algebra of multilinear alternate forms on  $L_M^x$  depending only on finite order jets.

The first and most important step in the work of Gelfand-Fuks was the complete determination of the cohomology  $H^*(\mathfrak{a}_n)$  of the topological Lie algebra of formal vector fields on  $R^n$ .

**THEOREM 1.** (Gelfand-Fuks [8], [9]). *Let  $E(h_1, \dots, h_n)$  be the exterior algebra on generators  $h_i$  of degree  $2i-1$  and let  $R[c_1, \dots, c_n]_{2n}^\wedge$  be the quotient of the polynomial algebra in generators  $c_i$  of degree  $2i$  by the ideal of elements of degree  $> 2n$ .*

Then a model for  $C^*(\mathfrak{a}_n)$  is the DG-algebra

$$WU_n = E(h_1, \dots, h_n) \otimes R[c_1, \dots, c_n] \hat{=}_{2n}$$

with  $dh_i = c_i$  and  $dc_i = 0$ .

It follows that  $H^i(\mathfrak{a}_n) = 0$  for  $1 \leq i \leq 2n$  and  $i > n^2 + 2n$ . Also the multiplicative structure is trivial; more precisely,  $WU_n$  is a model for a wedge of spheres (for instance  $S^3$  for  $n = 1$ ,  $S^5 \vee S^5 \vee S^7 \vee S^8 \vee S^8$  for  $n = 2$ ) (cf. Vey [9]).

$WU_n$  is also a model for the space  $F_n$  obtained by taking the restriction of the  $U_n$ -universal bundle over the  $2n$ -skeleton of its base space  $BU_n$  (cf. Gelfand-Fuks [8]). Note that this representation is compatible with the natural actions of  $O_n \subset U_n$ .

One can also consider the relative complex  $C^*(\mathfrak{a}_n, O_n)$  or  $C^*(\mathfrak{a}_n, SO_n)$  of  $O_n$  or  $SO_n$ -basic elements in  $C^*(\mathfrak{a}_n)$ , where  $O_n$  is the orthogonal group acting in the usual way on  $R^n$ , hence on  $\mathfrak{a}_n$ .

Define  $WO_n$  as the subalgebra of  $WU_n$  generated by the  $h_i$  with  $i$  odd and all the  $c_i$ . From theorem 1, it is easy to deduce the

THEOREM 1' [12].  $WO_n$  is a model for  $C^*(\mathfrak{a}_n, O_n)$ .

A model for  $C^*(\mathfrak{a}_n, SO_n)$  is  $WO_n$  for  $n$  odd and

$$WSO_n = WO_n \otimes R[e] / (e^2 - c_n)$$

for  $n$  even, where  $\deg e = n$  and  $de = 0$ .

From the finite dimensionality of  $H^*(\mathfrak{a}_n)$ , using a suitable spectral sequence, Gelfand and Fuks prove in particular [7].

THEOREM 2. If  $H^*(M)$  is finite dimensional, then  $H^*(L_M)$  is finite dimensional in each degree.

The Guillemin-Losik double complex.

First define  $C^*(L_M, \Omega_M)$  as the algebra of continuous alternate multilinear forms on  $L_M$  with values in the algebra  $\Omega_M$  of differential forms on  $M$ . We have two differentials, the first one defined as in 1 and the second one by the exterior differential in  $\Omega_M$ . So this is a double complex and we can consider the associated total differential.

$C_{\Delta}^*(L_M, \Omega_M)$  is the subcomplex of  $C^*(L_M, \Omega_M)$  of those forms associating to a sequence  $v_1, \dots, v_k$  of vector fields on  $M$  a differential form  $f(v_1, \dots, v_k)$  whose value at  $x \in M$  depends only on finite order jets of the  $v_i$ s at  $x$ .

THEOREM 3. (Guillemin [10], Losik [17]).  $C_{\Delta}^*(L_M, \Omega_M)$  is a model for a bundle  $E$  with fiber  $F_n$ , base space  $M$ , associated to the tangent bundle of  $M$ .

More precisely, a model for  $C_{\Delta}^*(L_M, \Omega_M)$  is the DG-algebra  $\Omega_M \otimes WU_n$  over  $\Omega_M$ , where

$$d(1 \otimes c_i) = 0 \quad d(1 \otimes h_i) = 1 \otimes c_i - p_{i/2} \otimes 1$$

where  $p_{i/2}$  is zero if  $i$  is odd and is a form representing the Pontrjagin class of  $M$  of degree  $2i$  if  $i$  is even.

Note that if a foliation  $F$  on  $X \times M$  transverse to the fibers  $\{x\} \times M$  is given, one has a characteristic homomorphism

$$C^*(L_M, \Omega_M) \rightarrow \Omega_{X \times M}$$

One has also a morphism

$$WO_n \rightarrow C_{\Delta}^*(L_M, \Omega_M)$$

(or  $WU_n \rightarrow C^*(L_M, \Omega_M)$  in case  $M$  has trivial Pontrjagin classes) whose composition with the previous one is the usual characteristic homomorphism for the foliation  $F$  (cf. [3], [12]).

#### 4. MAIN THEOREM

THEOREM 1.  $C^*(L_M)$  is a model for the space  $\Gamma$  of continuous sections of the bundle  $E$  described in the theorem above.

This result, first conjectured by Bott (and also Fuks), has been proved by several people (Bott-Segal<sup>1</sup>), Fuks-Segal, Haefliger [13], Ph. Trauber, and others).

Suppose that  $G$  is a compact connected Lie group acting on  $M$ . Then it also acts on the bundle  $E$  and on its space of sections. Let us denote by  $\Gamma_G$  the total space of the bundle with fiber  $\Gamma$  associated to the universal  $G$ -bundle with base space  $BG$ .

THEOREM 1'.  $C^*(L_M; G)$  is a model for the space  $\Gamma_G$ .

The way I proved theorem 1 was to construct first a tentative algebraic model  $A$  for  $\Gamma$  following ideas of R. Thom [20] and D. Sullivan [18], and

<sup>1</sup>) Added on proof: *Topology* 16 (1977), pp. 285-298.