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Autor: Haefliger, André
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through a map $H^*(BG; R) \rightarrow H^*(L_M; G)$ so that we get a commutative diagram

$$\begin{array}{ccc} & & H^*(L_M; G) \\ & \nearrow & \downarrow \\ H^*(BG; R) & & \\ & \searrow & \\ & & H^*(X; R) \end{array}$$

So it is important to compute the map $H^*(BG; R) \rightarrow H^*(L_M; G)$. When G is a compact connected Lie group, then $H^*(BG; R)$ is the algebra $I(G)$ of invariant polynomials on the Lie algebra of G , and the map from $I(G)$ to $C^*(L_M; G)$ is given by a G -connexion in $C^*(L_M)$ (cf. [5]).

In the example above, namely $M = S^1$ and $G = SO_2$, then $H^*(BSO_2)$ is a polynomial algebra in a generator of degree 2, the Euler class, which is mapped on a non zero multiple of e .

3. THE FORMAL VECTOR FIELDS AND THE DIAGONAL COMPLEX

Given a point x on M , we can consider the Lie algebra L_M^x of infinite jets at x of vector fields on M with the quotient topology. It is isomorphic to the Lie algebra \mathfrak{a}_n of formal vector fields $\sum v_i(x) \partial/\partial x^i$ in R^n , where the $v_i(x)$ are formal power series in the coordinates x^1, \dots, x^n .

The natural map $L_M \rightarrow L_M^x$ associating to a vector field its jet at x gives a DG -algebra morphism

$$C^*(L_M^x) \rightarrow C^*(L_M)$$

where $C^*(L_M^x)$ is the algebra of multilinear alternate forms on L_M^x depending only on finite order jets.

The first and most important step in the work of Gelfand-Fuks was the complete determination of the cohomology $H^*(\mathfrak{a}_n)$ of the topological Lie algebra of formal vector fields on R^n .

THEOREM 1. (Gelfand-Fuks [8], [9]). *Let $E(h_1, \dots, h_n)$ be the exterior algebra on generators h_i of degree $2i-1$ and let $R[c_1, \dots, c_n]_{2n}^\wedge$ be the quotient of the polynomial algebra in generators c_i of degree $2i$ by the ideal of elements of degree $> 2n$.*

Then a model for $C^*(\mathfrak{a}_n)$ is the DG-algebra

$$WU_n = E(h_1, \dots, h_n) \otimes R[c_1, \dots, c_n] \hat{=}_{2n}$$

with $dh_i = c_i$ and $dc_i = 0$.

It follows that $H^i(\mathfrak{a}_n) = 0$ for $1 \leq i \leq 2n$ and $i > n^2 + 2n$. Also the multiplicative structure is trivial; more precisely, WU_n is a model for a wedge of spheres (for instance S^3 for $n = 1$, $S^5 \vee S^5 \vee S^7 \vee S^8 \vee S^8$ for $n = 2$) (cf. Vey [9]).

WU_n is also a model for the space F_n obtained by taking the restriction of the U_n -universal bundle over the $2n$ -skeleton of its base space BU_n (cf. Gelfand-Fuks [8]). Note that this representation is compatible with the natural actions of $O_n \subset U_n$.

One can also consider the relative complex $C^*(\mathfrak{a}_n, O_n)$ or $C^*(\mathfrak{a}_n, SO_n)$ of O_n or SO_n -basic elements in $C^*(\mathfrak{a}_n)$, where O_n is the orthogonal group acting in the usual way on R^n , hence on \mathfrak{a}_n .

Define WO_n as the subalgebra of WU_n generated by the h_i with i odd and all the c_i . From theorem 1, it is easy to deduce the

THEOREM 1' [12]. WO_n is a model for $C^*(\mathfrak{a}_n, O_n)$.

A model for $C^*(\mathfrak{a}_n, SO_n)$ is WO_n for n odd and

$$WSO_n = WO_n \otimes R[e] / (e^2 - c_n)$$

for n even, where $\deg e = n$ and $de = 0$.

From the finite dimensionality of $H^*(\mathfrak{a}_n)$, using a suitable spectral sequence, Gelfand and Fuks prove in particular [7].

THEOREM 2. If $H^*(M)$ is finite dimensional, then $H^*(L_M)$ is finite dimensional in each degree.

The Guillemin-Losik double complex.

First define $C^*(L_M, \Omega_M)$ as the algebra of continuous alternate multilinear forms on L_M with values in the algebra Ω_M of differential forms on M . We have two differentials, the first one defined as in 1 and the second one by the exterior differential in Ω_M . So this is a double complex and we can consider the associated total differential.

$C_{\Delta}^*(L_M, \Omega_M)$ is the subcomplex of $C^*(L_M, \Omega_M)$ of those forms associating to a sequence v_1, \dots, v_k of vector fields on M a differential form $f(v_1, \dots, v_k)$ whose value at $x \in M$ depends only on finite order jets of the v_i s at x .

THEOREM 3. (Guillemin [10], Losik [17]). $C_{\Delta}^*(L_M, \Omega_M)$ is a model for a bundle E with fiber F_n , base space M , associated to the tangent bundle of M .

More precisely, a model for $C_{\Delta}^*(L_M, \Omega_M)$ is the DG-algebra $\Omega_M \otimes WU_n$ over Ω_M , where

$$d(1 \otimes c_i) = 0 \quad d(1 \otimes h_i) = 1 \otimes c_i - p_{i/2} \otimes 1$$

where $p_{i/2}$ is zero if i is odd and is a form representing the Pontrjagin class of M of degree $2i$ if i is even.

Note that if a foliation F on $X \times M$ transverse to the fibers $\{x\} \times M$ is given, one has a characteristic homomorphism

$$C^*(L_M, \Omega_M) \rightarrow \Omega_{X \times M}$$

One has also a morphism

$$WO_n \rightarrow C_{\Delta}^*(L_M, \Omega_M)$$

(or $WU_n \rightarrow C^*(L_M, \Omega_M)$ in case M has trivial Pontrjagin classes) whose composition with the previous one is the usual characteristic homomorphism for the foliation F (cf. [3], [12]).

4. MAIN THEOREM

THEOREM 1. $C^*(L_M)$ is a model for the space Γ of continuous sections of the bundle E described in the theorem above.

This result, first conjectured by Bott (and also Fuks), has been proved by several people (Bott-Segal¹), Fuks-Segal, Haefliger [13], Ph. Trauber, and others).

Suppose that G is a compact connected Lie group acting on M . Then it also acts on the bundle E and on its space of sections. Let us denote by Γ_G the total space of the bundle with fiber Γ associated to the universal G -bundle with base space BG .

THEOREM 1'. $C^*(L_M; G)$ is a model for the space Γ_G .

The way I proved theorem 1 was to construct first a tentative algebraic model A for Γ following ideas of R. Thom [20] and D. Sullivan [18], and

¹) Added on proof: *Topology* 16 (1977), pp. 285-298.