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through a map $H^* (BG; R) \to H^* (L_M; G)$ so that we get a commutative diagram

$$\begin{tikzcd} & H^*(L_M; G) \arrow[d] \\ H^*(BG; R) \arrow[ur] \arrow[r] & H^*(X; R) \end{tikzcd}$$

So it is important to compute the map $H^* (BG; R) \to H^* (L_M; G)$. When G is a compact connected Lie group, then $H^* (BG; R)$ is the algebra $I(G)$ of invariant polynomials on the Lie algebra of G, and the map from $I(G)$ to $C^* (L_M; G)$ is given by a G-connexion in $C^* (L_M)$ (cf. [5]).

In the example above, namely $M = S^1$ and $G = SO_2$, then $H^* (BSO_2)$ is a polynomial algebra in a generator of degree 2, the Euler class, which is mapped on a non zero multiple of e.

3. The formal vector fields and the diagonal complex

Given a point x on M, we can consider the Lie algebra L_M^x of infinite jets at x of vector fields on M with the quotient topology. It is isomorphic to the Lie algebra $\mathfrak{a}_n$ of formal vector fields $\sum v_i (x)\, \partial / \partial x^i$ in R^n, where the $v_i (x)$ are formal power series in the coordinates $x^1, ..., x^n$.

The natural map $L_M \to L_M^x$ associating to a vector field its jet at x gives a DG-algebra morphism

$$C^* (L_M^x) \to C^* (L_M)$$

where $C^* (L_M^x)$ is the algebra of multilinear alternate forms on L_M^x depending only on finite order jets.

The first and most important step in the work of Gelfand-Fuks was the complete determination of the cohomology $H^* (\mathfrak{a}_n)$ of the topological Lie algebra of formal vector fields on R^n.

Theorem 1. (Gelfand-Fuks [8], [9]). *Let $E(h_1, ..., h_n)$ be the exterior algebra on generators h_i of degree $2i-1$ and let $R[c_1, ..., c_n]^{\wedge}_{2n}$ be the quotient of the polynomial algebra in generators c_i of degree $2i$ by the ideal of elements of degree $> 2n$.*

Then a model for $C^*(\mathfrak{a}_n)$ *is the DG-algebra*

$$WU_n = E(h_1, \ldots, h_n) \otimes R[c_1, \ldots, c_n]\,\hat{}_{2n}$$

with $dh_i = c_i$ *and* $dc_i = 0$.

It follows that $H^i(\mathfrak{a}_n) = 0$ for $1 \leqslant i \leqslant 2n$ and $i > n^2 + 2n$. Also the multiplicative structure is trivial; more precisely, WU_n is a model for a wedge of spheres (for instance S^3 for $n = 1$, $S^5 \vee S^5 \vee S^7 \vee S^8 \vee S^8$ for $n = 2$) (cf. Vey [9]).

WU_n is also a model for the space F_n obtained by taking the restriction of the U_n-universal bundle over the $2n$-skeleton of its base space BU_n (cf. Gelfand-Fuks [8]). Note that this representation is compatible with the natural actions of $O_n \subset U_n$.

One can also consider the relative complex $C^*(\mathfrak{a}_n, O_n)$ or $C^*(\mathfrak{a}_n, SO_n)$ of O_n or SO_n-basic elements in $C^*(\mathfrak{a}_n)$, where O_n is the orthogonal group acting in the usual way on R^n, hence on $\mathfrak{a}_n$.

Define WO_n as the subalgebra of WU_n generated by the h_i with i odd and all the c_i. From theorem 1, it is easy to deduce the

THEOREM 1′ [12]. WO_n *is a model for* $C^*(\mathfrak{a}_n, O_n)$.
A model for $C^*(\mathfrak{a}_n, SO_n)$ *is* WO_n *for* n *odd and*

$$WSO_n = WO_n \otimes R[e]/(e^2 - c_n)$$

for n *even, where* $\deg e = n$ *and* $de = 0$.

From the finite dimensionality of $H^*(\mathfrak{a}_n)$, using a suitable spectral sequence, Gelfand and Fuks prove in particular [7].

THEOREM 2. *If* $H^*(M)$ *is finite dimensional, then* $H^*(L_M)$ *is finite dimensional in each degree.*

The Guillemin-Losik double complex.

First define $C^*(L_M, \Omega_M)$ as the algebra of continuous alternate multilinear forms on L_M with values in the algebra Ω_M of differential forms on M. We have two differentials, the first one defined as in 1 and the second one by the exterior differential in Ω_M. So this is a double complex and we can consider the associated total differential.

$C^*_\Delta(L_M, \Omega_M)$ is the subcomplex of $C^*(L_M, \Omega_M)$ of those forms associating to a sequence $v_1, \ldots, v_k$ of vector fields on M a differential form $f(v_1, \ldots, v_k)$ whose value at $x \in M$ depends only on finite order jets of the v_i's at x.

THEOREM 3. (Guillemin [10], Losik [17]). *$C^*_\Delta(L_M, \Omega_M)$ is a model for a bundle E with fiber F_n, base space M, associated to the tangent bundle of M.*

More precisely, a model for $C^*_\Delta(L_M, \Omega_M)$ is the DG-algebra $\Omega_M \otimes WU_n$ over Ω_M, where

$$d(1 \otimes c_i) = 0 \quad d(1 \otimes h_i) = 1 \otimes c_i - p_{i/2} \otimes 1$$

where $p_{i/2}$ is zero if i is odd and is a form representing the Pontrjagin class of M of degree $2i$ if i is even.

Note that if a foliation F on $X \times M$ transverse to the fibers $\{x\} \times M$ is given, one has a characteristic homomorphism

$$C^*(L_M, \Omega_M) \to \Omega_{X \times M}$$

One has also a morphism

$$WO_n \to C^*_\Delta(L_M, \Omega_M)$$

(or $WU_n \to C^*(L_M, \Omega_M)$ in case M has trivial Pontrjagin classes) whose composition with the previous one is the usual characteristic homomorphism for the foliation F (cf. [3], [12]).

4. MAIN THEOREM

THEOREM 1. *$C^*(L_M)$ is a model for the space Γ of continuous sections of the bundle E described in the theorem above.*

This result, first conjectured by Bott (and also Fuks), has been proved by several people (Bott-Segal [1]), Fuks-Segal, Haefliger [13], Ph. Trauber, and others).

Suppose that G is a compact connected Lie group acting on M. Then it also acts on the bundle E and on its space of sections. Let us denote by Γ_G the total space of the bundle with fiber Γ associated to the universal G-bundle with base space BG.

THEOREM 1'. *$C^*(L_M; G)$ is a model for the space Γ_G.*

The way I proved theorem 1 was to construct first a tentative algebraic model A for Γ following ideas of R. Thom [20] and D. Sullivan [18], and

[1]) Added on proof: *Topology 16* (1977), pp. 285-298.