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ON THE GELFAND-FUKS COHOMOLOGY [1]

by André HAEFLIGER

In this talk, we would like to report on the work of Gelfand and Fuks on the cohomology of the Lie algebra L_M of smooth vector fields on a manifold M, as well as on more recent developments, some of them obtained in collaboration with Raoul Bott.

1. DEFINITIONS

Gelfand-Fuks cohomology.

L_M will denote the Lie algebra of smooth vector fields on M, with the topology of uniform convergence of all derivatives on compact sets. For M compact, L_M can be thought as the Lie algebra of the group Diff_M of diffeomorphisms of M.

Gelfand and Fuks [7], have considered the differential graded algebra $C^*(L_M)$ of *continuous* multilinear alternate forms on L_M with values in R, the differential of a k-form f being the $(k+1)$-form df defined by

$$df(v_0, \ldots, v_k) = \sum_{o \leq r < s \leq k} (-1)^{r+s} f([v_r, v_s], v_0, \ldots, \hat{v}_r, \ldots, \hat{v}_s, \ldots, v_k)$$

where the v_i's are vector fields on M. So those cochains are like distributions.

Suppose that G is a Lie group acting smoothly and effectively on M. Then the Lie algebra $\mathfrak{g}$ of G is identified with a subalgebra of L_M. We shall denote by $C^*(L_M; G)$ the subalgebra of $C^*(L_M)$ of G-basic cochains, namely cochains invariant by G and which vanish if one of the argument v_i belongs to $\mathfrak{g}$.

The cohomology of $C^*(L_M)$ (resp. $C^*(L_M; G)$) will be denoted by $H^*(L_M)$ (resp. $H^*(L_M; G)$), and will be called the Gelfand-Fuks cohomology of M (resp. of M rel. to G).

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[1]) Presented at the Colloquium on Topology and Algebra, April 1977, Zurich

Models (Sullivan Theory) (cf. [18]).

$C^*(L_M)$ and $C^*(L_M; G)$ are examples of differential graded commutative (in the graded sense) algebras over **R**, abbreviated DG-algebras.

Among DG-algebras, we consider the equivalence relation generated by "$A \sim B$" if there is a morphism $\varphi: A \to B$ of DG-algebras inducing an isomorphism on cohomology. We say that M is a *model* for A if M is equivalent to A under this equivalence relation. Following the terminology of Sullivan, we say that M is a *minimal model* for A (assuming $H^o(A) = R$ and $H^1(A) = 0$) if M is a *free* algebra (namely the tensor product of a polynomial algebra on even dimensional generators by an exterior algebra on odd dimensional generators), the differential of each generator being decomposable (we also assume that generators are of degree bigger than one). The free algebra on a set of generators x_α will be denoted by $A(x_\alpha)$.

There is a contravariant functor from the category of topological spaces to the category of DG-algebras associating to the space X the DG-algebra $A^*(X)$ of real polynomial forms on its singular complex. If one takes instead rational polynomial forms, this functor induces an equivalence between rational homotopy types of 1-connected spaces with finite dimensional cohomology and equivalence classes of 1-connected DG algebras over Q with finite dimensional cohomology. A minimal model corresponds to a Postnikov decomposition. In particular the vector space of generators in the minimal model is the dual of the graded vector space $\pi_*(X) \otimes R$, where $\pi_i(X)$ is the i-th homotopy group of X.

We shall say that a DG-algebra A is a model for the space X if it is a model for the DG-algebra $A^*(X)$.

The main problem is to find good models for the DG-algebras $C^*(L_M)$ or $C^*(L_M; G)$, if possible finite dimensional in each degree.

As an example computed by Gelfand and Fuks [6], consider the case of the circle S^1. Then $H^*(L_{S^1})$ is the free algebra on generators u and v of degree 2 and 3 represented by the cocycles

$$u(f, g) = \int_0^1 \begin{vmatrix} f' & f'' \\ g' & g'' \end{vmatrix} dx \quad \text{and} \quad v(f, g, h) = \int \begin{vmatrix} f & f' & f'' \\ g & g' & g'' \\ h & h' & h'' \end{vmatrix} dx$$

where the vector fields on S^1 are identified with functions of period 1 on R. This is also a model for $C^*(L_{S^1})$.

If G is the group SO_2 of rotations of S^1, then $H(L_{S^1}; SO_2)$ is a model for $C^*(L_{S^1}; SO_2)$. It is generated by u and by an element e of degree 2 represented by

$$e(f, g) = \int_0^1 \begin{vmatrix} f, & g \\ f', & g' \end{vmatrix} dx$$

The only relation is $e\,u = 0$.

2. Connection with foliations

Let me indicate very briefly the relation with characteristic classes of flat bundles (cf. [12]).

$H^*(L_M, G)$ could also be interpreted as the differentiable cohomology of a suitable differentiable category (for more informations see [4] and [15]).

We consider on the product $X \times M$ of a smooth manifold X with M a smooth foliation F whose leaves have the same dimension as X and cut each fibers $\{x\} \times M$ transversally.

To such a foliation is naturally associated a continuous DG-algebra map

$$\chi_F : C^*(L_M) \to \Omega_X$$

where Ω_X is the DG-algebra of differential forms on X. In fact there is a bijection between such morphisms and foliations F as above.

Passing to cohomology, we get the characteristic map

$$H^*(L_M) \to H^*(X; R)$$

If we replace the trivial bundle by a bundle E with fiber M, base space X and structural group G, then for a foliation F on E complementary to the fibers, we still get a morphism

$$\chi_F : C^*(L_M; G) \to \Omega_X$$

hence a characteristic homomorphism

$$H^*(L_M, G) \to H^*(X; R)$$

Denoting by BG the classifying space for G-bundles, we also have the usual characteristic map $H^*(BG; R) \to H^*(X; R)$. This map factorizes