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\S 2. The Lefschetz trace formula for the c.f.p. index

This reduces to the classical Lefschetz-Hopf theorem if B = a point, or if p: E = B. Our assumptions in 2.1 are a little more restrictive than necessary, in order to facilitate the proof; a slight generalization is indicated in 2.8.

(2.1) THEOREM. Let $p: E \to B$ be an ENR_B , where B is a compact ENR. Let $g: D_g \to E, \varphi: D_{\varphi} \to B$ denote maps as in 1.1 such that Fix (g) is compact, and $D_{\varphi} \supset \text{Fix}(g)$. Then the c.f.p. index of (g, φ) agrees with the Lefschetz trace of the composite $hB \xrightarrow{\varphi} h$ Fix $(g) \xrightarrow{t} hB$, or $hB \xrightarrow{\varphi^*} hD \xrightarrow{t^D} hB$, where $t = t_g$ is the fixed-point transfer (cf. [2], § 3), D is any neighborhood of Fix (g) in D_{φ} , h is singular and h is Cech-cohomology with coefficients in \mathbb{Z} or \mathbb{Q} . In formulas,

(2.2)
$$J(g,\varphi) = tr(t_g \circ \varphi) = tr(t_g \circ \varphi^*).$$

Proof. Using a vertical neighborhood retraction we can assume that $E = \mathbf{R}^n \times B$; this is, in fact, what the definition 1.6-1.9 shows (if $Y = \mathbf{R}^n$). Then $g(y, b) = (\gamma(y, b), b)$, where $\gamma: D_g \to \mathbf{R}^n$, and $J(g, \varphi) = I(\gamma, \varphi)$ as explained in 1.12. Furthermore, since *B* is ENR, we have $\iota: B \subset U \subset \mathbf{R}^m$ and a retraction $\rho: U \to B$, where *U* is open in \mathbf{R}^m . We can then extend φ, γ, g to maps φ, γ, g of open subsets of $\mathbf{R}^n \times U \subset \mathbf{R}^n \times \mathbf{R}^m$ by composing with $id \times \rho: \mathbf{R}^n \times U \to \mathbf{R}^n \times B$. The fixed points of (γ, φ) , (γ, φ) (and their index) are the same, by commutativity [1], VIII, 5.16 — since $(\gamma, \varphi) = (id \times \iota) (\gamma, \varphi) (id \times \rho)$. Altogether (omitting the ~), we can assume that φ, γ, g are defined in open subsets $D_{\varphi}, D_{\gamma} = D_g$ of $\mathbf{R}^n \times U, \varphi: D_{\varphi} \to B \subset U, \gamma: D_g \to \mathbf{R}^n, D_{\varphi} \to \text{Fix } (g)$, Fix (g) is (no longer compact but) proper over *U*; in particular, $K = \text{Fix } (g) \cap (p^{-1}B)$ is compact.

We now argue in a similar (although simpler) fashion as on p. 241 of [2]. We consider the following diagram (explanations below).



Here, $\mathbf{R}_o^n = (\mathbf{R}^n, \mathbf{R}^n - 0)$, X is an open neighborhood of Fix (g) in which γ and φ are defined, $K = \text{Fix}(g) \cap (p^{-1}B)$, $X_B = X \cap (p^{-1}B)$, $q: X \subset \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is the projection, d(u, b) = u - b. The dotted arrows stand for sequences of inclusion maps (as in [2], 3.3); some of these go the wrong way but then they are homotopy equivalences or excisions, inducing isomorphisms in cohomology. For instance, α stands for

$$\mathbf{R}_{o}^{n} \times \mathbf{R}_{o}^{m} = \mathbf{R}_{o}^{n+m} \sim (\mathbf{R}^{n+m}, \mathbf{R}^{n+m} - C) \hookrightarrow (\mathbf{R}^{n+m}, \mathbf{R}^{n+m} - K) \xleftarrow{EXC} (X, X-K)$$

where C is a ball around 0, containing K. Similarly for j on the left. β is a relative version (compare [2], 3.7), namely

$$\mathbf{R}_{o}^{n} \times (U, U-B) \sim (\mathbf{R}^{n}, \mathbf{R}^{n} - C') \times (U, U-B) \subset \mathbf{Q}^{n} \times U, \quad (\mathbf{R}^{n} \times U - \operatorname{Fix}(g)) \cup (\mathbf{R}^{n} \times (U-B)) \xleftarrow{EXC} \mathbf{Q}^{n} \times (U-F) \to (X, (X - \operatorname{Fix}(g)) \cup (X - X_{B})),$$

where C' is a ball around $0 \in \mathbb{R}^n$ such that $K \subset (C' \times B)$. The lower t_g will be explained later.

The reader might want to follow the track of an element across the diagram 2.3; it looks as follows

We now apply cohomology $h = H^*(-; \mathbf{Q})$ to the diagram 2.3. Let $s^n \in h^n \mathbf{R}_o^n$ the canonical generator. Then $s^n \times s^m$ generates $h^{n+m}(\mathbf{R}_o^n \times \mathbf{R}_o^m)$, and its image along the top row of 2.3 is $I(g, \gamma) s^n \times s^m = J(g, \varphi) s^n \times s^m$, by definitions [1], VII, 5.2, and 1.9 above.

The left part of the lower row (which is marked t_g) induces the relative transfer (or trace) homomorphism $t_g: h(X, X-X_B) \rightarrow h(U, U-B)$, as defined in [2], 3.6-8. In formulas,

(2.4)
$$s^n \times \xi \ s^n \mapsto \times t_g^{X, Z}(\xi), \ Z = X - X_B.$$

Actually, [2], 3.8 is a little more general: it maps $h(X, X-X_B)$ into h(U, U), where $\tilde{U} \supset (U-B)$; we've composed [2], 3.8 with $h(U, \tilde{U}) \rightarrow h(U, U-B)$. Using the Künneth-formula we can write

(2.5)
$$d^*s^m = \sum_{\nu} \alpha_{\nu} \times \beta_{\nu}, \text{ with } \alpha_{\nu} \in h(U, U-B), \beta_{\nu} \in hB.$$

Following $\alpha_v \times \beta_v$ along the lower row of (2.3) gives

(2.6)
$$\alpha_{\nu} \times \beta_{\nu} \mapsto t_g \left(p^* \alpha_{\nu} \cup \varphi^* \beta_{\nu} \right) = \alpha_{\nu} \cup \left(t_g \varphi^* \beta_{\nu} \right),$$

the latter because t_g is a homomorphism of modules over h(U, U-B), by the relative version of [2], 3.20.

If we define $\kappa: h(U, U-B) \to \mathbf{Q}$ by $j^*(u) = \kappa(u) s^m$ (this corresponds to γ on p. 233, line 3⁻ of [2]), then $s^n \times \alpha_v \times \beta_v$ has image $\kappa(\alpha_v \cup t_g \varphi^* \beta_v) s^n \times s^m$ in the upper left corner of 2.3. On the other hand $\kappa(\alpha_v \cup t_g \varphi^* \beta_v)$ is the trace of the endomorphism

$$\xi \mapsto (-1)^{|\beta_{\nu}|} \beta_{\nu} \kappa (\alpha_{\nu} \cup t_{g} \varphi^{*} \xi) , \quad \xi \in hB ,$$

by [2], 6.7. It follows, that the image of $d^*s^m = \sum_{v} s^n \times \alpha_v \times \beta_v$ in the upper left corner is $s^n \times s^m$ -times the trace of

(2.7)
$$\xi \mapsto \sum_{\nu} (-1)^{|\beta_{\nu}|} \beta_{\nu} \kappa (\alpha_{\nu} \cup t_{g} \varphi^{*} \xi), \ \xi \in hB,$$

and so $J(g, \varphi) =$ trace of 2.7.

It remains to show that 2.7 agrees with $t_g^B \varphi_B^*$, where we now add indices (B, or U) to indicate the range of t_g resp. the domaine of φ^* . This will follow from [2], 6.16 which asserts (in greater generality) that $\sum_{v} (-1)^{|\beta_v|} \beta_v \kappa (\alpha_v \cup \eta)$ = $\iota^* \eta$, for $\eta \in hU$ and $\iota^* \colon hU \to hB$. Taking $\eta = t_g^U \varphi_U^* \xi$ we see that 2.7 agrees with $\xi \mapsto \iota^* t_U^g \varphi_U^* \xi = t_g^B \varphi_B^* \xi$, the latter by naturality ([2], 3.12) of t_g applied to ι .

(2.8) Remark. The assumption in 2.1 that B be compact can be weakened: It suffices that for some compact subset $R \subset B$ we have that $\text{Fix}(g)_R = \text{Fix}(g) \cap (p^{-1}R)$ is compact, and

$$\operatorname{im}(\varphi) \subset R$$
, $D_{\varphi} \supset \operatorname{Fix}(g)_R$.

Then the composite $h R \xrightarrow{\phi} h$ (Fix $(g)_R$) $\xrightarrow{t_g} h R$ is defined, has finite rank, and has Lefschetz trace equal to $J(g, \phi)$.

Our proof of 2.1 can be adapted to this more general situation. Or, by arguments as in [2], 8.6, one can slightly increase R in B, and decrease D_{φ} , such that the increased R is a compact ENR, and over (the increased) R the assumptions of 2.1 are satisfied; then 2.1 will imply the more general result above.

§ 3. Applications, Problems.

(3.1) Whether and how the trace formula 2.1 can be used depends mainly on one's knowledge of the transfer t_a . For instance, one knows that

- (i) $t_g p^* = I(g_b) =$ multiplication with the Hopf-index of $g_b: D_g \cap p^{-1}b \to p^{-1}b$ (in ordinary cohomology, *B* connected).
- (ii) $t_g: hD_g \to hB$ is induced by a stable map of B^+ into D_g^+ ; in particular, it commutes with stable cohomology operations.
- (iii) t_g is itself given by a trace-formula if $p: E \to B$ is a bundle with compact fibres which are totally non-cohomologous to zero.

We shall now illustrate (cf. 3.2, 3.3, 3.5) how these properties can be used.

(3.2) Suppose φ is homotopic to $\beta(p \mid D_{\varphi})$, for some $\beta: B \to B$. Then $t_g \varphi^* = t_g p^* \beta^* = I(g_b) \beta^*$, provided B is connected (cf. [2], 4.8). Therefore

$$J(g,\varphi) = tr(t_g\varphi^*) = I(g_b)tr(\beta^*) = I(g_b)I(\beta).$$

Geometrically, this result is very plausible: If $\varphi = \beta_p$ then Coinc (φ, p) consists of all fibres $D_{\varphi} \cap p^{-1}b$ with $b \in \text{Fix}(\beta)$. The "number" of these fibres is $I(\beta)$, and in every fibre the "number" of fixed points of g equals $I(g_b)$. — As the geometry suggests, the result holds under more general assumptions and can be proved directly from §1 (it doesn't seriously use 2.1).

As an illustration, the reader might look at the case where $p: E \to B$ is the tangent sphere-bundle of a compact Riemannian manifold B, and $\varphi = \varphi_t: E \to B$, $\varphi(x) = \exp(tx)$, for $t \in \mathbb{R}$. Clearly $\varphi \simeq \varphi_0 = p$, and Coinc $(\varphi, p) = \emptyset$ if |t| is small enough, $t \neq 0$. Hence, $0 = J(g, \varphi)$ $= I(g_b) I(id_B) = I(g_b) \chi(B)$, for all g. (For a direct proof of this result the reader should think of Fix $(g) \subset E$ as a manifold such that p | Fix (g)has degree $I(g_b)$).

(3.3) The definition [2], 3.3-4 shows that t_g is a composite of geometric homomorphisms (induced by continuous maps) and suspension isomor-