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§ 2. THE LEFSCHETZ TRACE FORMULA FOR THE C.F.P. INDEX

This reduces to the classical Lefschetz-Hopf theorem if $B =$ a point, or if $p: E = B$. Our assumptions in 2.1 are a little more restrictive than necessary, in order to facilitate the proof; a slight generalization is indicated in 2.8.

(2.1) THEOREM. *Let $p: E \rightarrow B$ be an ENR_B , where B is a compact ENR. Let $g: D_g \rightarrow E, \varphi: D_\varphi \rightarrow B$ denote maps as in 1.1 such that $\text{Fix}(g)$ is compact, and $D_\varphi \supset \text{Fix}(g)$. Then the c.f.p. index of (g, φ) agrees with the Lefschetz trace of the composite $hB \xrightarrow{\check{\varphi}} \check{h} \text{Fix}(g) \xrightarrow{t} hB$, or $hB \xrightarrow{\varphi^*} hD \xrightarrow{t^D} hB$, where $t = t_g$ is the fixed-point transfer (cf. [2], § 3), D is any neighborhood of $\text{Fix}(g)$ in D_φ , h is singular and \check{h} is Čech-cohomology with coefficients in \mathbf{Z} or \mathbf{Q} . In formulas,*

$$(2.2) \quad J(g, \varphi) = \text{tr}(t_g \circ \check{\varphi}) = \text{tr}(t_g^D \circ \varphi^*).$$

Proof. Using a vertical neighborhood retraction we can assume that $E = \mathbf{R}^n \times B$; this is, in fact, what the definition 1.6-1.9 shows (if $Y = \mathbf{R}^n$). Then $g(y, b) = (\gamma(y, b), b)$, where $\gamma: D_g \rightarrow \mathbf{R}^n$, and $J(g, \varphi) = I(\gamma, \varphi)$ as explained in 1.12. Furthermore, since B is ENR, we have $\iota: B \subset U \subset \mathbf{R}^m$ and a retraction $\rho: U \rightarrow B$, where U is open in \mathbf{R}^m . We can then extend φ, γ, g to maps $\tilde{\varphi}, \tilde{\gamma}, \tilde{g}$ of open subsets of $\mathbf{R}^n \times U \subset \mathbf{R}^n \times \mathbf{R}^m$ by composing with $id \times \rho: \mathbf{R}^n \times U \rightarrow \mathbf{R}^n \times B$. The fixed points of $(\gamma, \varphi), (\tilde{\gamma}, \tilde{\varphi})$ (and their index) are the same, by commutativity [1], VIII, 5.16 — since $(\tilde{\gamma}, \tilde{\varphi}) = (id \times \iota)(\gamma, \varphi)(id \times \rho)$. Altogether (omitting the $\tilde{}$), we can assume that φ, γ, g are defined in open subsets $D_\varphi, D_\gamma = D_g$ of $\mathbf{R}^n \times U, \varphi: D_\varphi \rightarrow B \subset U, \gamma: D_\gamma \rightarrow \mathbf{R}^n, D_\varphi \supset \text{Fix}(g), \text{Fix}(g)$ is (no longer compact but) proper over U ; in particular, $K = \text{Fix}(g) \cap (p^{-1}B)$ is compact.

We now argue in a similar (although simpler) fashion as on p. 241 of [2]. We consider the following diagram (explanations below).

$$\begin{array}{ccc}
 \mathbf{R}_0^n \times \mathbf{R}_0^m & \xrightarrow{\alpha} & (X, X - K) & \xrightarrow{(q-\gamma, p-\varphi)} & \mathbf{R}_0^n \times \mathbf{R}_0^m \\
 \downarrow id \times j & & \parallel & & \uparrow id \times d \\
 \mathbf{R}_0^n \times (U, U - B) & \xrightarrow{\beta} & (X, (X - \text{Fix}(g)) \cup (X - X_B)) & \xrightarrow{(q-\gamma, id)} & \mathbf{R}_0^n \times (X, X - X_B) & \xrightarrow{id \times (p, \varphi)} & \mathbf{R}_0^n \times (U, U - B) \times B \\
 & & \boxed{t_g} & & & &
 \end{array}$$

(2.3)

Here, $\mathbf{R}_o^n = (\mathbf{R}^n, \mathbf{R}^n - 0)$, X is an open neighborhood of $\text{Fix}(g)$ in which γ and φ are defined, $K = \text{Fix}(g) \cap (p^{-1}B)$, $X_B = X \cap (p^{-1}B)$, $q: X \subset \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^n$ is the projection, $d(u, b) = u - b$. The dotted arrows stand for sequences of inclusion maps (as in [2], 3.3); some of these go the wrong way but then they are homotopy equivalences or excisions, inducing isomorphisms in cohomology. For instance, α stands for

$$\mathbf{R}_o^n \times \mathbf{R}_o^m = \mathbf{R}_o^{n+m} \sim (\mathbf{R}^{n+m}, \mathbf{R}^{n+m} - C) \hookrightarrow (\mathbf{R}^{n+m}, \mathbf{R}^{n+m} - K) \xleftarrow{EXC} \supset (X, X-K)$$

where C is a ball around 0, containing K . Similarly for j on the left. β is a relative version (compare [2], 3.7), namely

$$\begin{aligned} \mathbf{R}_o^n \times (U, U-B) &\sim (\mathbf{R}^n, \mathbf{R}^n - C') \times (U, U-B) \hookrightarrow \\ (\mathbf{R}^n \times U, (\mathbf{R}^n \times U - \text{Fix}(g)) \cup (\mathbf{R}^n \times (U-B))) &\xleftarrow{EXC} \supset \\ (X, (X - \text{Fix}(g)) \cup (X - X_B)), & \end{aligned}$$

where C' is a ball around $0 \in \mathbf{R}^n$ such that $K \subset (C' \times B)$. The lower t_g will be explained later.

The reader might want to follow the track of an element across the diagram 2.3; it looks as follows

$$\begin{array}{ccccccc} (y, b) & \dashrightarrow & (y, b) & \xrightarrow{\quad\quad\quad} & (y - \gamma(y, b), b - \varphi(y, b)) & & \\ \downarrow & & \parallel & & \uparrow & & \\ (y, b) & \dashrightarrow & (y, b) & \xrightarrow{\quad\quad\quad} & (y - \gamma(y, b), y, b) & \xrightarrow{\quad\quad\quad} & (y - \gamma(y, b), b, \varphi(y, b)). \end{array}$$

We now apply cohomology $h = H^*(-; \mathbf{Q})$ to the diagram 2.3. Let $s^n \in h^n \mathbf{R}_o^n$ the canonical generator. Then $s^n \times s^m$ generates $h^{n+m}(\mathbf{R}_o^n \times \mathbf{R}_o^m)$, and its image along the top row of 2.3 is $I(g, \gamma) s^n \times s^m = J(g, \varphi) s^n \times s^m$, by definitions [1], VII, 5.2, and 1.9 above.

The left part of the lower row (which is marked t_g) induces the relative transfer (or trace) homomorphism $t_g: h(X, X - X_B) \rightarrow h(U, U - B)$, as defined in [2], 3.6-8. In formulas,

$$(2.4) \quad s^n \times \xi \mapsto s^n \times t_g^{X, Z}(\xi), \quad Z = X - X_B.$$

Actually, [2], 3.8 is a little more general: it maps $h(X, X - X_B)$ into $h(U, \tilde{U})$, where $\tilde{U} \supset (U - B)$; we've composed [2], 3.8 with $h(U, \tilde{U}) \rightarrow h(U, U - B)$.

Using the Künneth-formula we can write

$$(2.5) \quad d^*s^m = \sum_{\nu} \alpha_{\nu} \times \beta_{\nu}, \text{ with } \alpha_{\nu} \in h(U, U-B), \beta_{\nu} \in hB.$$

Following $\alpha_{\nu} \times \beta_{\nu}$ along the lower row of (2.3) gives

$$(2.6) \quad \alpha_{\nu} \times \beta_{\nu} \mapsto t_g(p^*\alpha_{\nu} \circ \varphi^*\beta_{\nu}) = \alpha_{\nu} \circ (t_g\varphi^*\beta_{\nu}),$$

the latter because t_g is a homomorphism of modules over $h(U, U-B)$, by the relative version of [2], 3.20.

If we define $\kappa: h(U, U-B) \rightarrow \mathbf{Q}$ by $j^*(u) = \kappa(u)s^m$ (this corresponds to γ on p. 233, line 3⁻ of [2]), then $s^n \times \alpha_{\nu} \times \beta_{\nu}$ has image $\kappa(\alpha_{\nu} \circ t_g\varphi^*\beta_{\nu})s^n \times s^m$ in the upper left corner of 2.3. On the other hand $\kappa(\alpha_{\nu} \circ t_g\varphi^*\beta_{\nu})$ is the trace of the endomorphism

$$\xi \mapsto (-1)^{|\beta_{\nu}|} \beta_{\nu} \kappa(\alpha_{\nu} \circ t_g\varphi^*\xi), \quad \xi \in hB,$$

by [2], 6.7. It follows, that the image of $d^*s^m = \sum_{\nu} s^n \times \alpha_{\nu} \times \beta_{\nu}$ in the upper left corner is $s^n \times s^m$ -times the trace of

$$(2.7) \quad \xi \mapsto \sum_{\nu} (-1)^{|\beta_{\nu}|} \beta_{\nu} \kappa(\alpha_{\nu} \circ t_g\varphi^*\xi), \quad \xi \in hB,$$

and so $J(g, \varphi) = \text{trace of 2.7.}$

It remains to show that 2.7 agrees with $t_g^B\varphi_B^*$, where we now add indices (B , or U) to indicate the range of t_g resp. the domain of φ^* . This will follow from [2], 6.16 which asserts (in greater generality) that $\sum_{\nu} (-1)^{|\beta_{\nu}|} \beta_{\nu} \kappa(\alpha_{\nu} \circ \eta) = i^*\eta$, for $\eta \in hU$ and $i^*: hU \rightarrow hB$. Taking $\eta = t_g^U\varphi_U^*\xi$ we see that 2.7 agrees with $\xi \mapsto i^*t_g^U\varphi_U^*\xi = t_g^B\varphi_B^*\xi$, the latter by naturality ([2], 3.12) of t_g applied to i . \square

(2.8) *Remark.* The assumption in 2.1 that B be compact can be weakened: It suffices that for some compact subset $R \subset B$ we have that $\text{Fix}(g)_R = \text{Fix}(g) \cap (p^{-1}R)$ is compact, and

$$\text{im}(\varphi) \subset R, \quad D_{\varphi} \supset \text{Fix}(g)_R.$$

Then the composite $\check{h}R \xrightarrow{\check{\varphi}} \check{h}(\text{Fix}(g)_R) \xrightarrow{t_g} \check{h}R$ is defined, has finite rank, and has Lefschetz trace equal to $J(g, \varphi)$.

Our proof of 2.1 can be adapted to this more general situation. Or, by arguments as in [2], 8.6, one can slightly increase R in B , and decrease D_φ , such that the increased R is a compact ENR, and over (the increased) R the assumptions of 2.1 are satisfied; then 2.1 will imply the more general result above.

§ 3. APPLICATIONS, PROBLEMS.

(3.1) Whether and how the trace formula 2.1 can be used depends mainly on one's knowledge of the transfer t_g . For instance, one knows that

- (i) $t_g p^* = I(g_b) =$ multiplication with the Hopf-index of $g_b: D_g \cap p^{-1}b \rightarrow p^{-1}b$ (in ordinary cohomology, B connected).
- (ii) $t_g: hD_g \rightarrow hB$ is induced by a stable map of B^+ into D_g^+ ; in particular, it commutes with stable cohomology operations.
- (iii) t_g is itself given by a trace-formula if $p: E \rightarrow B$ is a bundle with compact fibres which are totally non-cohomologous to zero.

We shall now illustrate (cf. 3.2, 3.3, 3.5) how these properties can be used.

(3.2) Suppose φ is homotopic to $\beta(p \mid D_\varphi)$, for some $\beta: B \rightarrow B$. Then $t_g \varphi^* = t_g p^* \beta^* = I(g_b) \beta^*$, provided B is connected (cf. [2], 4.8). Therefore

$$J(g, \varphi) = \text{tr}(t_g \varphi^*) = I(g_b) \text{tr}(\beta^*) = I(g_b) I(\beta).$$

Geometrically, this result is very plausible: If $\varphi = \beta_p$ then $\text{Coinc}(\varphi, p)$ consists of all fibres $D_\varphi \cap p^{-1}b$ with $b \in \text{Fix}(\beta)$. The "number" of these fibres is $I(\beta)$, and in every fibre the "number" of fixed points of g equals $I(g_b)$. — As the geometry suggests, the result holds under more general assumptions and can be proved directly from § 1 (it doesn't seriously use 2.1).

As an illustration, the reader might look at the case where $p: E \rightarrow B$ is the tangent sphere-bundle of a compact Riemannian manifold B , and $\varphi = \varphi_t: E \rightarrow B$, $\varphi(x) = \exp(tx)$, for $t \in \mathbf{R}$. Clearly $\varphi \simeq \varphi_0 = p$, and $\text{Coinc}(\varphi, p) = \emptyset$ if $|t|$ is small enough, $t \neq 0$. Hence, $0 = J(g, \varphi) = I(g_b) I(id_B) = I(g_b) \chi(B)$, for all g . (For a direct proof of this result the reader should think of $\text{Fix}(g) \subset E$ as a manifold such that $p \mid \text{Fix}(g)$ has degree $I(g_b)$).

(3.3) The definition [2], 3.3-4 shows that t_g is a composite of geometric homomorphisms (induced by continuous maps) and suspension isomor-