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## § 2. THE LEFSCHETZ TRACE FORMULA FOR THE C.F.P. INDEX

This reduces to the classical Lefschetz-Hopf theorem if  $B = \text{a point}$ , or if  $p: E = B$ . Our assumptions in 2.1 are a little more restrictive than necessary, in order to facilitate the proof; a slight generalization is indicated in 2.8.

(2.1) THEOREM. *Let  $p: E \rightarrow B$  be an  $ENR_B$ , where  $B$  is a compact ENR. Let  $g: D_g \rightarrow E$ ,  $\varphi: D_\varphi \rightarrow B$  denote maps as in 1.1 such that  $\text{Fix}(g)$  is compact, and  $D_\varphi \supset \text{Fix}(g)$ . Then the c.f.p. index of  $(g, \varphi)$  agrees with the Lefschetz trace of the composite  $hB \xrightarrow{\check{\varphi}} \check{h} \text{Fix}(g) \xrightarrow{t} hB$ , or  $hB \xrightarrow{\varphi^*} hD \xrightarrow{t^D} hB$ , where  $t = t_g$  is the fixed-point transfer (cf. [2], § 3),  $D$  is any neighborhood of  $\text{Fix}(g)$  in  $D_\varphi$ ,  $h$  is singular and  $\check{h}$  is Čech-cohomology with coefficients in  $\mathbf{Z}$  or  $\mathbf{Q}$ . In formulas,*

$$(2.2) \quad J(g, \varphi) = \text{tr}(t_g \circ \check{\varphi}) = \text{tr}(t_g^D \circ \varphi^*).$$

*Proof.* Using a vertical neighborhood retraction we can assume that  $E = \mathbf{R}^n \times B$ ; this is, in fact, what the definition 1.6-1.9 shows (if  $Y = \mathbf{R}^n$ ). Then  $g(y, b) = (\gamma(y, b), b)$ , where  $\gamma: D_g \rightarrow \mathbf{R}^n$ , and  $J(g, \varphi) = I(\gamma, \varphi)$  as explained in 1.12. Furthermore, since  $B$  is ENR, we have  $\iota: B \subset U \subset \mathbf{R}^m$  and a retraction  $\rho: U \rightarrow B$ , where  $U$  is open in  $\mathbf{R}^m$ . We can then extend  $\varphi, \gamma, g$  to maps  $\tilde{\varphi}, \tilde{\gamma}, \tilde{g}$  of open subsets of  $\mathbf{R}^n \times U \subset \mathbf{R}^n \times \mathbf{R}^m$  by composing with  $\text{id} \times \rho: \mathbf{R}^n \times U \rightarrow \mathbf{R}^n \times B$ . The fixed points of  $(\gamma, \varphi)$ ,  $(\tilde{\gamma}, \tilde{\varphi})$  (and their index) are the same, by commutativity [1], VIII, 5.16 — since  $(\tilde{\gamma}, \tilde{\varphi}) = (\text{id} \times \iota)(\gamma, \varphi)(\text{id} \times \rho)$ . Altogether (omitting the  $\sim$ ), we can assume that  $\varphi, \gamma, g$  are defined in open subsets  $D_\varphi, D_\gamma = D_g$  of  $\mathbf{R}^n \times U$ ,  $\varphi: D_\varphi \rightarrow B \subset U$ ,  $\gamma: D_\gamma \rightarrow \mathbf{R}^n$ ,  $D_\varphi \supset \text{Fix}(g)$ ,  $\text{Fix}(g)$  is (no longer compact but) proper over  $U$ ; in particular,  $K = \text{Fix}(g) \cap (p^{-1}B)$  is compact.

We now argue in a similar (although simpler) fashion as on p. 241 of [2]. We consider the following diagram (explanations below).

$$\begin{array}{ccccc}
\mathbf{R}_0^n \times \mathbf{R}_0^m & \xrightarrow{\alpha} & (X, X - K) & \xrightarrow{(q-\gamma, p-\varphi)} & \mathbf{R}_0^n \times \mathbf{R}_0^m \\
\downarrow id \times j & & \parallel & & \downarrow id \times d \\
\mathbf{R}_0^n \times (U, U - B) & \xrightarrow{\beta} & (X, (X - \text{Fix}(g)) \cup (X - X_B)) & \xrightarrow{(q-\gamma, id)} & \mathbf{R}_0^n \times (X, X - X_B) & \xrightarrow{id \times (p, \varphi)} & \mathbf{R}_0^n \times (U, U - B) \times B \\
& & \underbrace{\hspace{10em}}_{t_g} & & & & 
\end{array}$$

Here,  $\mathbf{R}_o^n = (\mathbf{R}^n, \mathbf{R}^n - 0)$ ,  $X$  is an open neighborhood of  $\text{Fix}(g)$  in which  $\gamma$  and  $\varphi$  are defined,  $K = \text{Fix}(g) \cap (p^{-1}B)$ ,  $X_B = X \cap (p^{-1}B)$ ,  $q: X \subset \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^n$  is the projection,  $d(u, b) = u - b$ . The dotted arrows stand for sequences of inclusion maps (as in [2], 3.3); some of these go the wrong way but then they are homotopy equivalences or excisions, inducing isomorphisms in cohomology. For instance,  $\alpha$  stands for

$$\mathbf{R}_o^n \times \mathbf{R}_o^m = \mathbf{R}_o^{n+m} \sim (\mathbf{R}^{n+m}, \mathbf{R}^{n+m} - C) \hookrightarrow (\mathbf{R}^{n+m}, \mathbf{R}^{n+m} - K) \xleftarrow{EXC} (X, X-K)$$

where  $C$  is a ball around 0, containing  $K$ . Similarly for  $j$  on the left.  $\beta$  is a relative version (compare [2], 3.7), namely

$$\begin{aligned} \mathbf{R}_o^n \times (U, U-B) &\sim (\mathbf{R}^n, \mathbf{R}^n - C') \times (U, U-B) \hookrightarrow \\ &(\mathbf{R}^n \times U, (\mathbf{R}^n \times U - \text{Fix}(g)) \cup (\mathbf{R}^n \times (U-B))) \xleftarrow{EXC} \\ &(X, (X - \text{Fix}(g)) \cup (X - X_B)), \end{aligned}$$

where  $C'$  is a ball around  $0 \in \mathbf{R}^n$  such that  $K \subset (C' \times B)$ . The lower  $t_g$  will be explained later.

The reader might want to follow the track of an element across the diagram 2.3; it looks as follows

$$\begin{array}{ccccc} (y, b) & \dashrightarrow & (y, b) & \xrightarrow{\quad} & (y - \gamma(y, b), b - \varphi(y, b)) \\ \downarrow & & \parallel & & \uparrow \\ (y, b) & \dashrightarrow & (y, b) & \xrightarrow{\quad} & (y - \gamma(y, b), y, b) \xrightarrow{\quad} (y - \gamma(y, b), b, \varphi(y, b)). \end{array}$$

We now apply cohomology  $h = H^*(-; \mathbf{Q})$  to the diagram 2.3. Let  $s^n \in h^n \mathbf{R}_o^n$  the canonical generator. Then  $s^n \times s^m$  generates  $h^{n+m}(\mathbf{R}_o^n \times \mathbf{R}_o^m)$ , and its image along the top row of 2.3 is  $I(g, \gamma) s^n \times s^m = J(g, \varphi) s^n \times s^m$ , by definitions [1], VII, 5.2, and 1.9 above.

The left part of the lower row (which is marked  $t_g$ ) induces the relative transfer (or trace) homomorphism  $t_g: h(X, X - X_B) \rightarrow h(U, U - B)$ , as defined in [2], 3.6-8. In formulas,

$$(2.4) \quad s^n \times \xi \mapsto s^n \mapsto \times t_g^{X, Z}(\xi), \quad Z = X - X_B.$$

Actually, [2], 3.8 is a little more general: it maps  $h(X, X - X_B)$  into  $h(U, \tilde{U})$ , where  $\tilde{U} \supset (U - B)$ ; we've composed [2], 3.8 with  $h(U, \tilde{U}) \rightarrow h(U, U - B)$ .

Using the Künneth-formula we can write

$$(2.5) \quad d^*s^m = \sum_v \alpha_v \times \beta_v, \text{ with } \alpha_v \in h(U, U-B), \beta_v \in hB.$$

Following  $\alpha_v \times \beta_v$  along the lower row of (2.3) gives

$$(2.6) \quad \alpha_v \times \beta_v \mapsto t_g(p^*\alpha_v \cup \varphi^*\beta_v) = \alpha_v \cup (t_g\varphi^*\beta_v),$$

the latter because  $t_g$  is a homomorphism of modules over  $h(U, U-B)$ , by the relative version of [2], 3.20.

If we define  $\kappa: h(U, U-B) \rightarrow \mathbf{Q}$  by  $j^*(u) = \kappa(u)s^m$  (this corresponds to  $\gamma$  on p. 233, line 3<sup>-</sup> of [2]), then  $s^n \times \alpha_v \times \beta_v$  has image  $\kappa(\alpha_v \cup t_g\varphi^*\beta_v)s^n \times s^m$  in the upper left corner of 2.3. On the other hand  $\kappa(\alpha_v \cup t_g\varphi^*\beta_v)$  is the trace of the endomorphism

$$\xi \mapsto (-1)^{|\beta_v|} \beta_v \kappa(\alpha_v \cup t_g\varphi^*\xi), \quad \xi \in hB,$$

by [2], 6.7. It follows, that the image of  $d^*s^m = \sum_v s^n \times \alpha_v \times \beta_v$  in the upper left corner is  $s^n \times s^m$ -times the trace of

$$(2.7) \quad \xi \mapsto \sum_v (-1)^{|\beta_v|} \beta_v \kappa(\alpha_v \cup t_g\varphi^*\xi), \quad \xi \in hB,$$

and so  $J(g, \varphi) = \text{trace of 2.7.}$

It remains to show that 2.7 agrees with  $t_g^B\varphi_B^*$ , where we now add indices ( $B$ , or  $U$ ) to indicate the range of  $t_g$  resp. the domain of  $\varphi^*$ . This will follow from [2], 6.16 which asserts (in greater generality) that  $\sum_v (-1)^{|\beta_v|} \beta_v \kappa(\alpha_v \cup \eta) = \iota^*\eta$ , for  $\eta \in hU$  and  $\iota^*: hU \rightarrow hB$ . Taking  $\eta = t_g^U\varphi_U^*\xi$  we see that 2.7 agrees with  $\xi \mapsto \iota^*t_g^U\varphi_U^*\xi = t_g^B\varphi_B^*\xi$ , the latter by naturality ([2], 3.12) of  $t_g$  applied to  $\iota$ .  $\square$

(2.8) *Remark.* The assumption in 2.1 that  $B$  be compact can be weakened: It suffices that for some compact subset  $R \subset B$  we have that  $\text{Fix}(g)_R = \text{Fix}(g) \cap (p^{-1}R)$  is compact, and

$$\text{im}(\varphi) \subset R, \quad D_\varphi \supset \text{Fix}(g)_R.$$

Then the composite  $\check{h}R \xrightarrow{\check{\varphi}} \check{h}(\text{Fix}(g)_R) \xrightarrow{t_g} \check{h}R$  is defined, has finite rank, and has Lefschetz trace equal to  $J(g, \varphi)$ .

Our proof of 2.1 can be adapted to this more general situation. *Or*, by arguments as in [2], 8.6, one can slightly increase  $R$  in  $B$ , and decrease  $D_\varphi$ , such that the increased  $R$  is a compact ENR, and over (the increased)  $R$  the assumptions of 2.1 are satisfied; then 2.1 will imply the more general result above.

### § 3. APPLICATIONS, PROBLEMS.

(3.1) Whether and how the trace formula 2.1 can be used depends mainly on one's knowledge of the transfer  $t_g$ . For instance, one knows that

- (i)  $t_g p^* = I(g_b) =$  multiplication with the Hopf-index of  $g_b: D_g \cap p^{-1}b \rightarrow p^{-1}b$  (in ordinary cohomology,  $B$  connected).
- (ii)  $t_g: hD_g \rightarrow hB$  is induced by a stable map of  $B^+$  into  $D_g^+$ ; in particular, it commutes with stable cohomology operations.
- (iii)  $t_g$  is itself given by a trace-formula if  $p: E \rightarrow B$  is a bundle with compact fibres which are totally non-cohomologous to zero.

We shall now illustrate (cf. 3.2, 3.3, 3.5) how these properties can be used.

(3.2) Suppose  $\varphi$  is homotopic to  $\beta(p|D_\varphi)$ , for some  $\beta: B \rightarrow B$ . Then  $t_g \varphi^* = t_g p^* \beta^* = I(g_b) \beta^*$ , provided  $B$  is connected (cf. [2], 4.8). Therefore

$$J(g, \varphi) = \text{tr}(t_g \varphi^*) = I(g_b) \text{tr}(\beta^*) = I(g_b) I(\beta).$$

Geometrically, this result is very plausible: If  $\varphi = \beta_p$  then  $\text{Coinc}(\varphi, p)$  consists of all fibres  $D_\varphi \cap p^{-1}b$  with  $b \in \text{Fix}(\beta)$ . The "number" of these fibres is  $I(\beta)$ , and in every fibre the "number" of fixed points of  $g$  equals  $I(g_b)$ . — As the geometry suggests, the result holds under more general assumptions and can be proved directly from § 1 (it doesn't seriously use 2.1).

As an illustration, the reader might look at the case where  $p: E \rightarrow B$  is the tangent sphere-bundle of a compact Riemannian manifold  $B$ , and  $\varphi = \varphi_t: E \rightarrow B$ ,  $\varphi(x) = \exp(tx)$ , for  $t \in \mathbf{R}$ . Clearly  $\varphi \simeq \varphi_0 = p$ , and  $\text{Coinc}(\varphi, p) = \emptyset$  if  $|t|$  is small enough,  $t \neq 0$ . Hence,  $0 = J(g, \varphi) = I(g_b) I(id_B) = I(g_b) \chi(B)$ , for all  $g$ . (For a direct proof of this result the reader should think of  $\text{Fix}(g) \subset E$  as a manifold such that  $p|_{\text{Fix}(g)}$  has degree  $I(g_b)$ ).

(3.3) The definition [2], 3.3-4 shows that  $t_g$  is a composite of geometric homomorphisms (induced by continuous maps) and suspension isomor-