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§ 1. THE COINCIDENCE-FIXED-POINT (C.F.P.) INDEX

(1.1) Let  $p: E \rightarrow B$  denote a euclidean neighborhood retract over  $B$  (abbrev.  $\text{ENR}_B$ ), where  $B$ , and hence  $E$ , is an ENR. Altogether this means that  $p: E \rightarrow B$  embeds as a neighborhood retract into the projection  $\mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^m$ , for some  $m, n$ . We refer the reader to [2], §1, for the precise definitions but remark that every smooth submersion and every fibration (with base and total space ENR) qualifies for  $p: E \rightarrow B$ .

We consider continuous maps  $g: D_g \rightarrow E$ ,  $\varphi: D_\varphi \rightarrow B$ , where  $D_g, D_\varphi$  are open subsets of  $E$ , and  $pg = p|_{D_g}$  (i.e.,  $g$  is fibre-preserving). We let  $\text{Fix}(g) = \{x \in D_g \mid gx = x\}$  and  $\text{Coinc}(\varphi, p) = \{x \in D_\varphi \mid \varphi x = px\}$ , and we assume that  $\text{Fix}(g) \cap \text{Coinc}(\varphi, p)$  is compact. Under these circumstances we shall define an integer  $J(g, \varphi) \in \mathbf{Z}$  which is akin to the Hopf fixed-point index. It “counts” the points in  $\text{Fix}(g) \cap \text{Coinc}(\varphi, p)$  in a weighted and homotopy-invariant fashion. It is the Hopf index of  $g$  resp.  $\varphi$  if  $B$  is a single point resp.  $p$  is the identity map of  $B$ .

(1.2) By definition [2], 1.1 of an  $\text{ENR}_B$ , we have that  $E$  is a fibre-preserving neighborhood retract of some  $\mathbf{R}^n \times B$ . In fact, for the present purpose we can use any product  $Y \times B$ , i.e. we'll use mappings  $E \xrightarrow{i} V \xrightarrow{r} E$  such that  $V \subset Y \times B$  is open,  $ri = id$ , and  $i, r$  are maps over  $B$ . In formulas,

$$(1.3) \quad ix = (i'x, px), \quad \text{where } i': E \rightarrow Y,$$

$$(1.4) \quad pr(y, b) = b, \quad \text{for } (y, b) \in V,$$

$$(1.5) \quad r(i'x, px) = x, \quad \text{for } x \in E.$$

Consider the following sequence of maps

$$(1.6) \quad D_g \cap D_\varphi \xrightarrow{(g, \varphi)} E \times B \xrightarrow{i' \times id} Y \times B \supset V \xrightarrow{r} E.$$

Its composite  $[g, \varphi]$  is defined in  $D_V = (i'g, \varphi)^{-1} V$  which is an open subset of  $(D_g \cap D_\varphi)$ , and hence of  $E$ . Thus

$$(1.7) \quad [g, \varphi]: D_V \rightarrow E, \quad [g, \varphi](x) = r(i'gx, \varphi x).$$

If  $x \in D_g \cap \text{Coinc}(\varphi, p)$  then

$$(i'g, \varphi)x = (i'gx, px) = (i'gx, pgx) = igx \in V,$$

hence  $[g, \varphi]x$  is defined and equals  $rigx = gx$ . It follows that  $\text{Fix}(g) \cap \text{Coinc}(\varphi, p) \subset \text{Fix}[g, \varphi] = \{x \in D_V \mid [g, \varphi]x = x\}$ . Conversely,  $x$

$= [g, \varphi] x$  implies  $px = p [g, \varphi] x = pr (i'gx, \varphi x) = \varphi x$ , hence  $x \in \text{Coinc}(\varphi, p)$ , and  $gx = [g, \varphi] x = x$ . Altogether

$$(1.8) \quad \text{Fix}(g) \cap \text{Coinc}(\varphi, p) = \text{Fix}[g, \varphi].$$

In particular,  $[g, \varphi]: D_V \rightarrow E$  has a compact fixed-point set, and we can assign to it its Hopf-index  $I[g, \varphi] \in \mathbf{Z}$  — for instance as in [1], VII,5.10. Furthermore,

(1.9) PROPOSITION AND DEFINITION. *The Hopf-index  $I[g, \varphi] \in \mathbf{Z}$  depends only on  $(g, \varphi)$ , not on the choice of the neighborhood retraction  $i, r$ . We denote this integer by  $J(g, \varphi)$ , and call it the *c.f.p.-index* of  $(g, \varphi)$ ; thus  $J(g, \varphi) = I[g, \varphi]$ .*

*Proof.* Because the range  $B$  of the maps  $\varphi, p$  is ENR, these two maps are homotopic in a neighborhood of  $\text{Coinc}(\varphi, p)$ . In fact (cf. [1], IV,8.6), there is an open neighborhood  $U$  of  $\text{Coinc}(\varphi, p)$  in  $D_\varphi$ , and a deformation  $\vartheta_t: U \rightarrow B, 0 \leq t \leq 1$ , such that

$$(1.10) \quad \vartheta_0 = p|U, \vartheta_1 = \varphi|U, \vartheta_t x = px \text{ for } x \in \text{Coinc}(\varphi, p) \text{ and all } t.$$

Consider then two neighborhood retractions

$$\begin{aligned} E &\xrightarrow{i} V \xrightarrow{r} E, \quad V \subset Y \times B; \quad ix = (i'x, px), \\ E &\xrightarrow{j} W \xrightarrow{s} E, \quad W \subset Z \times B; \quad jx = (j'x, px), \end{aligned}$$

as above, and the corresponding maps  $[g, \varphi]_1, [g, \varphi]_2$  as defined by 1.6. We have to show  $I([g, \varphi]_1) = I([g, \varphi]_2)$ . In order to do so we can (cf. [1], VII,5.11) restrict attention to an arbitrary open neighborhood  $N$  of  $\text{Fix}([g, \varphi]_i) = \text{Fix}(g) \cap \text{Coinc}(\varphi, p)$ . And we shall show that  $[g, \varphi]_i|N$  are homotopic ( $i=1, 2$ ) without moving the fixed point set, provided  $N$  is sufficiently small. The homotopy is given by the formula

$$(1.11) \quad \theta_t x = s(j'r(i'gx, \vartheta_t x), \varphi x).$$

This is defined for  $(x, t)$  such that  $x \in D_g \cap U, v = (i'gx, \vartheta_t x) \in V$ , and  $w = (j'rv, \varphi x) \in W$ ; the set of all such  $(x, t)$  is an open subset  $D_\theta$  of  $E \times [0, 1]$ . If  $x \in \text{Fix}(g) \cap \text{Coinc}(\varphi, p)$  then

$$v = (i'gx, \vartheta_t x) = (i'x, px) = ix \in V, \text{ and } rv = x,$$

hence

$$w = (j'rv, \varphi x) = (j'x, px) = jx \in W, \text{ and } \theta_t x = sw = x.$$

Therefore,  $(\text{Fix}(g) \cap \text{Coinc}(\varphi, p)) \times [0, 1] \subset D_\theta$ , and  $(\text{Fix}(g) \cap \text{Coinc}(\varphi, p)) \subset \text{Fix}(\theta_t)$  for all  $t$ . It follows that

$$N = \{x \in E \mid (x, t) \in D_\theta \text{ for all } t\}$$

is an open neighborhood of  $\text{Fix}(g) \cap \text{Coinc}(\varphi, p)$  in which the deformation  $\theta$  is defined (by 1.11).

Suppose now  $x \in N$  is a fixed point of  $\theta_t$ , thus  $x = s(j'r(i'gx, \mathfrak{D}_t x), \varphi x)$ . Apply  $p$ , using 1.4 for  $s$ , and get  $px = \varphi x =$ , hence  $\mathfrak{D}_t x = px$  by 1.10, hence  $r(i'gx, \mathfrak{D}_t x) = r(i'gx, px) = r(i'gx, pgx) = rigx = gx$ , hence  $x = \theta_t x = s(j'gx, px) = s(j'gx, pgx) = sjgx = gx$ ; altogether,  $x \in \text{Coinc}(\varphi, p) \cap \text{Fix}(g)$ . It follows that the fixed point set  $\text{Fix}(\theta_t) = \text{Fix}(g) \cap \text{Coinc}(\varphi, p)$  for all  $t$ . In particular,  $\cup_{t \in [0, 1]} \text{Fix}(\theta_t)$  is compact, hence (cf. [1].VII,5.15) all  $\theta_t$  have the same Hopf-index  $I(\theta_t)$ . But  $r(i'gx, \mathfrak{D}_0 x) = r(i'gx, px) = r(i'gx, pgx) = gx$ , hence  $\theta_0 x = s(j'gx, \varphi x) = [g, \varphi]_2 x$ . To calculate  $\theta_1$  we first remark that  $p[g, \varphi]_1 x = \varphi x$ , by 1.7 and 1.4; also  $r(i'gx, \mathfrak{D}_1 x) = r(i'gx, \varphi x) = [g, \varphi]_1 x$ , hence  $\theta_1 x = s(j'[g, \varphi]_1 x, p[g, \varphi]_1 x) = sj[g, \varphi]_1 x = [g, \varphi]_1 x$ .  $\square$

(1.12) *The product case*  $E = F \times B$ ,  $p = \text{projection}$ . In this case  $g: D_g \rightarrow F \times B$  has the form  $g(y, b) = (\gamma(y, b), b)$  with  $\gamma: D_g \rightarrow F$ . The two maps  $(\gamma, \varphi)$  combine to a map  $(\gamma, \varphi): D \rightarrow F \times B$ , where  $D (= D_g \cap D_\varphi)$  is an open subset of  $F \times B$ , and  $\text{Fix}(\gamma, \varphi) = \text{Fix}(g) \cap \text{Coinc}(\varphi, p)$ . In order to obtain the c.f.p.-index  $J(g, \varphi)$  one can use  $Y = F$  and the neighborhood retraction  $i = r = \text{identity-map}$  of  $Y \times B$ . The definition 1.9 then shows that

$$J(g, \varphi) = I(\gamma, \varphi);$$

i.e. *in the product case the c.f.p.-index of  $(g, \varphi)$  is simply the Hopf-index of  $(y, b) \mapsto (\gamma(y, b), \varphi(y, b))$ .*

The procedure 1.6-1.9 in the general case, on the other hand, can be considered as a reduction to the product case.

(1.13) *General properties of  $J(g, \varphi)$*  follow from corresponding properties of the Hopf-index. For instance,  $J(g, \varphi)$  is additive with respect to topological-sum decompositions of  $\text{Fix}(g) \cap \text{Coinc}(g, \varphi)$ , it is invariant under deformations such that  $\cup_{0 \leq t \leq 1} \text{Fix}(g_t) \cap \text{Coinc}(\varphi_t, p)$  is compact, it depends only on the germ of  $(g, \varphi)$  around  $\text{Fix}(g) \cap \text{Coinc}(\varphi, p)$  — in particular,  $J(g, \varphi) = 0$  if  $\text{Fix}(g) \cap \text{Coinc}(\varphi, p) = \emptyset$ , etc. These details are left to the reader. Lefschetz-trace formulas for  $J(g, \varphi)$  can be found in 2.1 and 3.5.