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2. APPLICATION TO THE PONCELET PROBLEM

We consider two smooth conics C and D meeting transversely at four points x_i ($i=0, 1, 2, 3$) of the projective plane \mathbf{P}^2 . The dual conic $D^* \subset \mathbf{P}^{2*}$ consists of the tangent lines ξ to D , and we consider the incidence correspondence

$$E \subset C \times D^*$$

of pairs $p = (x, \xi)$ with $x \in \xi$ (c.f. Figure 1 above). E is the basic algebraic curve underlying the Poncelet construction, and we shall now examine it.

Referring again to Figure 1, there are on E a pair of involutions defined by

$$\begin{cases} i(x, \xi) = (x', \xi) \\ i'(x', \xi) = (x', \xi') \end{cases}$$

whose composition $j = i' \circ i$ is given by $j(x, \xi) = (x', \xi')$. It follows that Poncelet's construction beginning at $p = (x, \xi)$ gives a closed polygon of n sides if, and only if,

$$j^n(p) = p.$$

The mapping

$$(x, \xi) \rightarrow x$$

represents $E \rightarrow C$ as a two-sheeted branched covering whose branch points are just the points $x_i \in C \cap D$ ($i=0, 1, 2, 3$), and the involution i' interchanges the sheets of this mapping (c.f. Figure 2 below). Similarly, i interchanges the two sheets of the mapping $E \rightarrow D^*$ given by $(x, \xi) \rightarrow \xi$ whose branch points are the four bitangents to the pair of conics. It follows that if we choose the origin to be $\mathfrak{o} = (x_0, \xi_0)$ in Figure 2 below

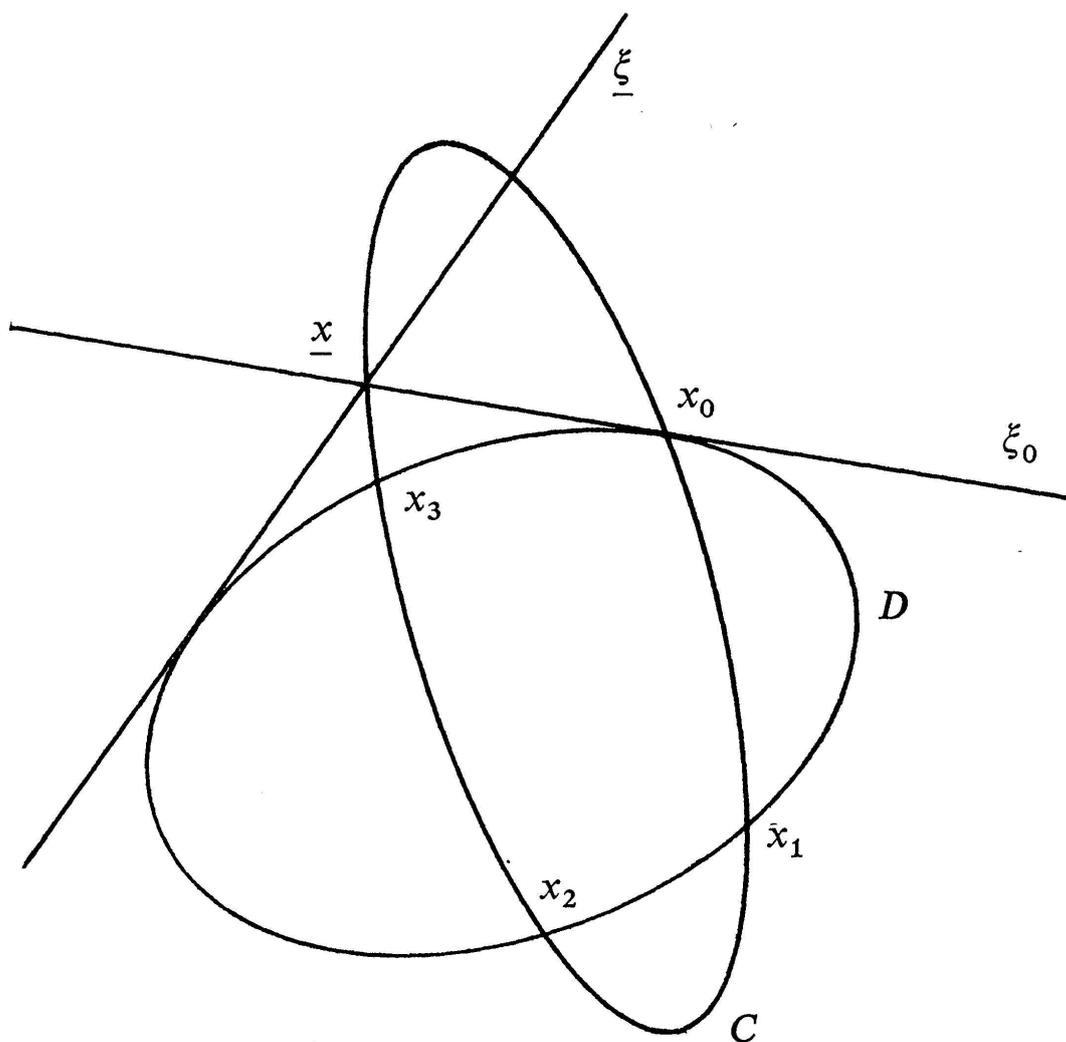


Figure 2

then E is an elliptic curve; i.e. a smooth algebraic curve of genus one with a marked point chosen as the identity for the group law. If we let $p = (\underline{x}, \underline{\xi})$ in Figure 2, then the Poncelet theorem is:

The Poncelet construction gives a closed polygon of n sides with arbitrary initial data $q = (x, \xi) \in E$ if, and only if,

$$(11) \quad np = \mathfrak{o}$$

on the elliptic curve E .

Proof. We want to show that (11) is equivalent to

$$j^n(q) = q$$

for an arbitrary point $q \in E$. On the universal covering C of E any involution i_1 having at least one fixed point lifts to

$$\tilde{i}_1(u) \equiv -u + v \text{ modulo } \Lambda,$$

and $i_1(\mathfrak{o}) = \mathfrak{o}$ is equivalent to $v \in \Lambda$. It follows that

$$\begin{cases} \tilde{i}(u) \equiv -u - w \text{ modulo } \Lambda \\ \tilde{i}'(u) \equiv -u \text{ modulo } \Lambda, \end{cases}$$

so that

$$\tilde{j}(u) \equiv u + w \text{ modulo } \Lambda,$$

and consequently

$$j^n(q) = q \Leftrightarrow nw \equiv 0 \text{ modulo } \Lambda.$$

Taking p to be the image of w in $E = \mathbf{C}/\Lambda$, we have

$$p = j(\mathfrak{o}) = (\underline{x}, \underline{\xi})$$

in Figure 2, which proves our assertion. Q.E.D.

To complete our story we want to combine this result with the explicit formula (10). As in the introduction we consider the pencil of conics

$$D_t = \{ t C(x) + D(x) = 0 \}$$

passing through the four base points x_i . The determinant $\det(t C(x) + D(x))$ is a cubic polynomial in t with non-zero roots $t_i (i=1, 2, 3)$. For $t \neq t_i$ we draw the tangent line to D_t through x_0 meeting C in a unique residual point $x(t)$. It is easy to see that $t = t_i$ is mapped into x_i (with suitable indexing), and since $D_\infty = C$ the value $t = \infty$ is mapped to x_0 . Taking $t = 0$ we see that $t = 0$ corresponds to x , so that in summary:

The elliptic curve E is birationally equivalent to the Riemann surface of the algebraic function $\sqrt{\det(t C(x) + D(x))}$ with the origin \mathfrak{o} corresponding to $t = \infty$ and the point $p = (\underline{x}, \underline{\xi})$ to one of the two points lying over $t = 0$.

Combining this with (10) gives Cayley's result stated in the introduction.

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