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1. POINTS OF FINITE ORDER ON ELLIPTIC CURVES

Let E be an elliptic curve over the complex numbers with origin \mathfrak{o} . In practice E will have various realizations as an algebraic curve defined by polynomial equations in projective space; e.g., as a plane cubic, the intersection of two quadrics in \mathbf{P}^3 , etc. All of these projective models are birationally isomorphic to the given curve E . It is well known that E admits a commutative group law with \mathfrak{o} being the identity, and we are interested in the points p of finite order n defined by

$$np = \mathfrak{o}$$

where $np = p + \dots + p$ (n times). Specifically, we pose the question of finding a projective model of E relative to which these points have a simple explicit description.

From a complex-analytic point of view we may realize E as the Riemann surface

$$E = \mathbf{C}/\Lambda$$

obtained by factoring the complex u -plane by a lattice Λ with $u = 0$ projecting onto the origin \mathfrak{o} ; this is a consequence of Abel's theorem¹⁾. The group law on E is obtained from the additive structure on \mathbf{C} , and so if $u_0 \in \mathbf{C}$ projects onto $p \in E$ the finite order condition is

$$(1) \quad nu_0 \equiv 0 \pmod{\Lambda}.$$

In particular there are n^2 points of finite order n on E corresponding to the points of

$$\frac{1}{n} \Lambda.$$

Our problem may be generalized to that of giving projective meaning to the equation

$$(2) \quad u_1 + \dots + u_n \equiv 0 \pmod{\Lambda},$$

which specializes to (1) when the u_i tend together. Here again the basic step is the following variant of *Abel's theorem*²⁾: *Given* $u_i, v_i \in \mathbf{C}$ ($i = 1, \dots, n$)

¹⁾ This is the classical version of Abel's theorem used in ¹⁾.

²⁾ C.f. L. Ahlfors, *Complex Analysis*, McGraw-Hill (New York), Exercise 2 on page 267. This may be thought of as providing a converse to the classical Abel's theorem.

there is an entire meromorphic function $f(u)$ with period lattice Λ and having zeroes at $u_i + \Lambda$ and poles at $v_i + \Lambda$ if, and only if,

$$u_1 + \dots + u_n \equiv v_1 + \dots + v_n \pmod{\Lambda}.$$

It follows that the vector space $H^0(\mathcal{O}_E([n\mathfrak{v}]))$ of rational functions on E having a pole of order at most n at \mathfrak{v} , or equivalently the entire meromorphic functions $f(u)$ which have period lattice Λ and a pole of order at most n at $u = 0$, has dimension n . If we choose a basis f_1, \dots, f_n for this vector space, then for $n \geq 3$ the mapping

$$F(u) = [f_1(u), \dots, f_n(u)]$$

induces a projective embedding

$$E \rightarrow \mathbf{P}^{n-1}$$

whose image is easily proved to be a smooth algebraic curve of degree n . Thus, for $n = 3$ we have a plane cubic, for $n = 4$ the intersection of two quadrics in \mathbf{P}^3 , etc. In general we shall call the image the *normal elliptic curve of degree n* . According to Abel's theorem the hyperplane sections of this curve, which are just the zeroes of functions $f \in H^0(\mathcal{O}_E([n\mathfrak{v}]))$, are characterized by $u_1 + \dots + u_n \equiv 0 \pmod{\Lambda}$. Put differently, the condition (2) is equivalent to

$$(3) \quad \det \|f_i(u_j)\| = 0$$

expressing the failure of the points $F(u_1), \dots, F(u_n)$ to be in general position. If we denote by

$$WF(u) = \begin{vmatrix} f_1(u) & \dots & f_n(u) \\ f'_1(u) & & f'_n(u) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ f_1^{(n-1)}(u) & \dots & f_n^{(n-1)}(u) \end{vmatrix}$$

the Wronskian of the functions $f_i(u)$, then by letting the u_i tend together the condition (3) specializes to the equation

$$(4) \quad WF(u) = 0$$

characterizing the solutions to (1). Points satisfying (4) will be called *hyperflexes*, and what we have shown is that:

The points of order n on an elliptic curve are precisely the hyperflexes of the normal elliptic curve of degree n .

Now we observe that the equation (4) is independent of the selection of basis $\{f_i\}$ and local coordinate u on E . To see therefore whether or not a given point p is of finite order n we will make convenient choices. Namely, we may choose a basis $\{1, f(u)\}$ for $H^0(\mathcal{O}_E([2\mathfrak{o}]))$ such that $f(p) = 0$. In other words, the function f induces a 2-to-1 map

$$(5) \quad f: E \rightarrow \mathbf{P}^1$$

with $p \in f^{-1}(0)$. It is well-known that the representation (5) has four branch points, one of which is the point at infinity with $f^{-1}(\infty) = \mathfrak{o}$. If we let x be the coordinate on \mathbf{P}^1 and a, b, c the finite branch points, then E is conformally represented as the Riemann surface of the algebraic function $\sqrt{(x-a)(x-b)(x-c)}$.

Put another way, the plane cubic curve with affine equation

$$(6) \quad y^2 = (x-a)(x-b)(x-c)$$

gives a projective model of E . Setting $x = f(u)$, since the holomorphic differential du is a constant multiple of dx/y it follows that, with a suitable

normalization, $2y = f'(u) = \frac{df(u)}{du}$. Consequently the projective model

(6) of E is given by the mapping $E \rightarrow \mathbf{P}^2$ associated to the basis $\{1, f(u), f'(u)\}$ of $H^0(\mathcal{O}_E([3\mathfrak{o}]))$. Of course, $f(u)$ and $f'(u)$ are essentially the Weierstrass functions. We recall that their Laurent series around $u = 0$ are

$$(7) \quad \left\{ \begin{array}{l} f(u) = \frac{1}{u^2} + \dots \\ f'(u) = \frac{-2}{u^3} + \dots \\ \vdots \\ f^{(k)}(u) = \frac{(-1)^k(k+1)!}{u^{k+2}} + \dots \end{array} \right.$$

Returning to our question of whether $p \in f^{-1}(0)$ is of finite order n , we will use $x = f(u)$ as local coordinate around p and choose the functions

$$(8) \quad \begin{cases} 1, x, \dots, x^m; y, xy, \dots, x^{m-1}y & n = 2m+1 \\ 1, x, \dots, x^m; y, xy, \dots, x^{m-2}y & n = 2m \end{cases}$$

as basis for $H^0(\mathcal{O}_E([n\mathfrak{d}]))$. That this choice gives a basis follows from the Laurent series (7). It is now an easy matter to express the Wronskian equation (4) at $x = 0$.

We consider the case $n = 2m + 1$ and let $\frac{dg(x)}{dx}$ be the derivative of $g(x)$ evaluated at $x = 0$. The choice of basis (8) facilitates the evaluation of the Wronskian. For example, from $\frac{d^k(x^l)}{dx^k} = 0$ for $k > l$ the Wronskian has the form

$$\begin{array}{c|c|c} 1 \dots 0 & \hline & \hline \\ \dots & \hline & \hline \\ \dots & \hline & \hline \\ \dots & \hline & \hline \\ 0 \dots m! & \hline & \hline \end{array} \quad , \quad \begin{array}{c|c|c} 0 \dots 0 & \hline & \hline \\ \dots & \hline & \hline \\ \dots & \hline & \hline \\ \dots & \hline & \hline \\ 0 \dots 0 & \hline & \hline \end{array}$$

so that (4) is equivalent to

$$(9) \quad \begin{array}{c|c} \frac{d^{m+1}y}{dx^{m+1}} & \frac{d^{m+1}(xy)}{dx^{m+1}} \\ \frac{d^{m+2}y}{dx^{m+2}} & \frac{d^{m+2}(xy)}{dx^{m+2}} \\ \vdots & \vdots \\ \frac{d^{2m}y}{dx^{2m}} & \frac{d^{2m}(xy)}{dx^{2m}} \end{array} \dots \begin{array}{c|c} \frac{d^{m+1}(x^{m-1}y)}{dx^{m+1}} \\ \frac{d^{m+2}(x^{m-1}y)}{dx^{m+2}} \\ \vdots \\ \frac{d^{2m}(x^{m-1}y)}{dx^{2m}} \end{array} = 0$$

If the series expansion of $y(x)$ is

$$y(x) = \sum_{k=0}^{\infty} A_k x^k,$$

then (9) is

$$\begin{vmatrix} (m+1)! A_{m+1} & (m+1)! A_m & \dots & (m+1)! A_2 \\ (m+2)! A_{m+2} & (m+2)! A_{m+1} & \dots & (m+2)! A_3 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ (2m)! A_{2m} & (2m)! A_{2m-1} & \dots & (2m)! A_{m+1} \end{vmatrix} = 0.$$

In summary we have proved

(10) Let E be an elliptic curve with origin \mathfrak{o} and $p \in E$ a given point. Then p is of finite order $n \Leftrightarrow$ the following condition is satisfied: Choose rational functions x, y on E having poles of respective orders 2, 3 at \mathfrak{o} but which are regular elsewhere and with $x(p) = 0$. Then there is an equation $y^2 = (x-a)(x-b)(x-c)$ where a, b, c are distinct and non-zero, and we write

$$y = \sqrt{(x-a)(x-b)(x-c)} = \sum_{k=0}^{\infty} A_k x^k.$$

The finite order condition is

$$\begin{vmatrix} A_2 & A_3 & \dots & A_{m+1} \\ A_3 & A_4 & \dots & A_{m+2} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ A_{m+1} & A_{m+2} & \dots & A_{2m} \end{vmatrix} = 0, \quad n = 2m + 1$$

$$\begin{vmatrix} A_3 & A_4 & \dots & A_{m+1} \\ A_4 & A_5 & \dots & A_{m+2} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ A_{m+1} & A_{m+2} & \dots & A_{2m} \end{vmatrix} = 0, \quad n = 2n.$$