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**Autor:** Griffiths, Phillip / Harris, Joseph  
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# 1. POINTS OF FINITE ORDER ON ELLIPTIC CURVES

Let  $E$  be an elliptic curve over the complex numbers with origin  $\mathfrak{o}$ . In practice  $E$  will have various realizations as an algebraic curve defined by polynomial equations in projective space; e.g., as a plane cubic, the intersection of two quadrics in  $\mathbf{P}^3$ , etc. All of these projective models are birationally isomorphic to the given curve  $E$ . It is well known that  $E$  admits a commutative group law with  $\mathfrak{o}$  being the identity, and we are interested in the points  $p$  of finite order  $n$  defined by

$$np = \mathfrak{o}$$

where  $np = p + \dots + p$  ( $n$  times). Specifically, we pose the question of finding a projective model of  $E$  relative to which these points have a simple explicit description.

From a complex-analytic point of view we may realize  $E$  as the Riemann surface

$$E = \mathbf{C}/\Lambda$$

obtained by factoring the complex  $u$ -plane by a lattice  $\Lambda$  with  $u = 0$  projecting onto the origin  $\mathfrak{o}$ ; this is a consequence of Abel's theorem<sup>1)</sup>. The group law on  $E$  is obtained from the additive structure on  $\mathbf{C}$ , and so if  $u_0 \in \mathbf{C}$  projects onto  $p \in E$  the finite order condition is

$$(1) \quad nu_0 \equiv 0 \text{ modulo } \Lambda.$$

In particular there are  $n^2$  points of finite order  $n$  on  $E$  corresponding to the points of

$$\frac{1}{n} \Lambda.$$

Our problem may be generalized to that of giving projective meaning to the equation

$$(2) \quad u_1 + \dots + u_n \equiv 0 \text{ modulo } \Lambda,$$

which specializes to (1) when the  $u_i$  tend together. Here again the basic step is the following variant of *Abel's theorem*<sup>2)</sup>: Given  $u_i, v_i \in \mathbf{C}$  ( $i = 1, \dots, n$ )

<sup>1)</sup> This is the classical version of Abel's theorem used in <sup>1)</sup>.

<sup>2)</sup> C.f. L. Ahlfors, *Complex Analysis*, McGraw-Hill (New York), Exercise 2 on page 267. This may be thought of as providing a converse to the classical Abel's theorem.

there is an entire meromorphic function  $f(u)$  with period lattice  $\Lambda$  and having zeroes at  $u_i + \Lambda$  and poles at  $v_i + \Lambda$  if, and only if,

$$u_1 + \dots + u_n \equiv v_1 + \dots + v_n \text{ modulo } \Lambda.$$

It follows that the vector space  $H^0(\mathcal{O}_E([n\mathfrak{o}]))$  of rational functions on  $E$  having a pole of order at most  $n$  at  $\mathfrak{o}$ , or equivalently the entire meromorphic functions  $f(u)$  which have period lattice  $\Lambda$  and a pole of order at most  $n$  at  $u = 0$ , has dimension  $n$ . If we choose a basis  $f_1, \dots, f_n$  for this vector space, then for  $n \geq 3$  the mapping

$$F(u) = [f_1(u), \dots, f_n(u)]$$

induces a projective embedding

$$E \rightarrow \mathbf{P}^{n-1}$$

whose image is easily proved to be a smooth algebraic curve of degree  $n$ . Thus, for  $n = 3$  we have a plane cubic, for  $n = 4$  the intersection of two quadrics in  $\mathbf{P}^3$ , etc. In general we shall call the image the *normal elliptic curve of degree  $n$* . According to Abel's theorem the hyperplane sections of this curve, which are just the zeroes of functions  $f \in H^0(\mathcal{O}_E([n\mathfrak{o}]))$ , are characterized by  $u_1 + \dots + u_n \equiv 0$  modulo  $\Lambda$ . Put differently, the condition (2) is equivalent to

$$(3) \quad \det \|f_i(u_j)\| = 0$$

expressing the failure of the points  $F(u_1), \dots, F(u_n)$  to be in general position. If we denote by

$$WF(u) = \begin{vmatrix} f_1(u) & \dots & f_n(u) \\ f'_1(u) & & f'_n(u) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ f_1^{(n-1)}(u) & \dots & f_n^{(n-1)}(u) \end{vmatrix}$$

the Wronskian of the functions  $f_i(u)$ , then by letting the  $u_i$  tend together the condition (3) specializes to the equation

$$(4) \quad WF(u) = 0$$

characterizing the solutions to (1). Points satisfying (4) will be called *hyperflexes*, and what we have shown is that:

*The points of order  $n$  on an elliptic curve are precisely the hyperflexes of the normal elliptic curve of degree  $n$ .*

Now we observe that the equation (4) is independent of the selection of basis  $\{f_i\}$  and local coordinate  $u$  on  $E$ . To see therefore whether or not a given point  $p$  is of finite order  $n$  we will make convenient choices. Namely, we may choose a basis  $\{1, f(u)\}$  for  $H^0(\mathcal{O}_E([2\mathfrak{o}]))$  such that  $f(p) = 0$ . In other words, the function  $f$  induces a 2-to-1 map

$$(5) \quad f: E \rightarrow \mathbf{P}^1$$

with  $p \in f^{-1}(0)$ . It is well-known that the representation (5) has four branch points, one of which is the point at infinity with  $f^{-1}(\infty) = \mathfrak{o}$ . If we let  $x$  be the coordinate on  $\mathbf{P}^1$  and  $a, b, c$  the finite branch points, then  $E$  is conformally represented as the Riemann surface of the algebraic function  $\sqrt{(x-a)(x-b)(x-c)}$ .

Put another way, the plane cubic curve with affine equation

$$(6) \quad y^2 = (x-a)(x-b)(x-c)$$

gives a projective model of  $E$ . Setting  $x = f(u)$ , since the holomorphic differential  $du$  is a constant multiple of  $dx/y$  it follows that, with a suitable normalization,  $2y = f'(u) = \frac{df(u)}{du}$ . Consequently the projective model

(6) of  $E$  is given by the mapping  $E \rightarrow \mathbf{P}^2$  associated to the basis  $\{1, f(u), f'(u)\}$  of  $H^0(\mathcal{O}_E([3\mathfrak{o}]))$ . Of course,  $f(u)$  and  $f'(u)$  are essentially the Weierstrass functions. We recall that their Laurent series around  $u = 0$  are

$$(7) \quad \left\{ \begin{array}{l} f(u) = \frac{1}{u^2} + \dots \\ f'(u) = \frac{-2}{u^3} + \dots \\ \cdot \\ \cdot \\ \cdot \\ f^{(k)}(u) = \frac{(-1)^k (k+1)!}{u^{k+2}} + \dots \end{array} \right.$$

Returning to our question of whether  $p \in f^{-1}(0)$  is of finite order  $n$ , we will use  $x = f(u)$  as local coordinate around  $p$  and choose the functions

$$(8) \quad \begin{cases} 1, x, \dots, x^m; y, xy, \dots, x^{m-1}y & n = 2m + 1 \\ 1, x, \dots, x^m; y, xy, \dots, x^{m-2}y & n = 2m \end{cases}$$

as basis for  $H^0(\mathcal{O}_E([n\mathfrak{o}]))$ . That this choice gives a basis follows from the Laurent series (7). It is now an easy matter to express the Wronskian equation (4) at  $x = 0$ .

We consider the case  $n = 2m + 1$  and let  $\frac{dg(x)}{dx}$  be the derivative of  $g(x)$  evaluated at  $x = 0$ . The choice of basis (8) facilitates the evaluation of the Wronskian. For example, from  $\frac{d^k(x^l)}{dx^k} = 0$  for  $k > l$  the Wronskian has the form

$$\begin{vmatrix} 1 & \dots & 0 & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ 0 & \dots & m! & & \\ \hline 0 & \dots & 0 & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ 0 & \dots & 0 & & \end{vmatrix},$$

so that (4) is equivalent to

$$(9) \quad \begin{vmatrix} \frac{d^{m+1}y}{dx^{m+1}} & \frac{d^{m+1}(xy)}{dx^{m+1}} & \dots & \frac{d^{m+1}(x^{m-1}y)}{dx^{m+1}} \\ \frac{d^{m+2}y}{dx^{m+2}} & \frac{d^{m+2}(xy)}{dx^{m+2}} & \dots & \frac{d^{m+2}(x^{m-1}y)}{dx^{m+2}} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{d^{2m}y}{dx^{2m}} & \frac{d^{2m}(xy)}{dx^{2m}} & \dots & \frac{d^{2m}(x^{m-1}y)}{dx^{2m}} \end{vmatrix} = 0$$

If the series expansion of  $y(x)$  is

$$y(x) = \sum_{k=0}^{\infty} A_k x^k,$$

then (9) is

$$\begin{vmatrix} (m+1)! A_{m+1} & (m+1)! A_m & \dots & (m+1)! A_2 \\ (m+2)! A_{m+2} & (m+2)! A_{m+1} & \dots & (m+2)! A_3 \\ & \cdot & & \cdot \\ & \cdot & & \cdot \\ & \cdot & & \cdot \\ (2m)! A_{2m} & (2m)! A_{2m-1} & \dots & (2m)! A_{m+1} \end{vmatrix} = 0.$$

In summary we have proved

- (10) Let  $E$  be an elliptic curve with origin  $\mathfrak{o}$  and  $p \in E$  a given point. Then  $p$  is of finite order  $n \Leftrightarrow$  the following condition is satisfied: Choose rational functions  $x, y$  on  $E$  having poles of respective orders 2, 3 at  $\mathfrak{o}$  but which are regular elsewhere and with  $x(p) = 0$ . Then there is an equation  $y^2 = (x-a)(x-b)(x-c)$  where  $a, b, c$  are distinct and non-zero, and we write

$$y = \sqrt{(x-a)(x-b)(x-c)} = \sum_{k=0}^{\infty} A_k x^k.$$

The finite order condition is

$$\begin{vmatrix} A_2 & A_3 & \dots & A_{m+1} \\ A_3 & A_4 & \dots & A_{m+2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_{m+1} & A_{m+2} & \dots & A_{2m} \end{vmatrix} = 0, \quad n = 2m + 1$$

$$\begin{vmatrix} A_3 & A_4 & \dots & A_{m+1} \\ A_4 & A_5 & \dots & A_{m+2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_{m+1} & A_{m+2} & \dots & A_{2m} \end{vmatrix} = 0, \quad n = 2n.$$