15. Cohomology of Algebraic Systems

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unions. In modern language, it gave for stable homotopy Π_*^S a spectral sequence $H_* (X, \Pi_*^S Y) \Rightarrow \Pi_*^S (X * Y)$, where $X * Y$ is the join of the spaces X and Y. In discussion with Adams, Whitehead talks about his definition of a generalized homology theory K and said that his paper "should" have proved $H_* (X, K_* (\mathrm{pt})) \Rightarrow K_* (X)$. Later, Atiyah told Adams about his joint work with Hirzebruch on K-theory as a generalized cohomology; he also wondered about its relation to ordinary cohomology. Adams, recalling the words of Whitehead, observed that there was a suitable spectral sequence; Atiyah asked how it was constructed and whether it was published. Adams thus reported that it was constructed in the inevitable way, from an appropriate filtration—but that it had not been published. Atiyah resigned himself to the trouble of writing it up—and so it is now called the Atiyah-Hirzebruch sequence. Given the familiarity at that time with the technique of spectral sequences, it is clear that this sequence was sure to be discovered at about that time—if not by one author, then by another.

15. Cohomology of Algebraic Systems

The cohomology of groups was just the starting point for the study of corresponding cohomology theorems of other sorts of algebraic systems. A few months after the discovery of the cohomology of groups, Hochschild found a corresponding cohomology for algebras. Again, the 2-dimensional cohomology group of an algebra corresponded to an extension problem for algebras, and it soon turned out that the Eilenberg-Mac Lane interpretation of H^3 as obstructions for non-abelian extensions of groups could also be carried over to algebras. Presently Chevalley and Eilenberg formulated a cohomology theory for Lie algebras. It was now amply clear that the idea of cohomology, originally conceived as a measure of the connectivity of spaces, was also relevant as a record of some of the aspects of quite a variety of algebraic systems. The connection with topology remained strong, however. For example, the Eilenberg-Mac Lane spaces $K(\Pi, n)$ were defined topologically, as spaces with Π the only non-vanishing homotopy group- in dimension n; their stable cohomology, however, could be interpreted as the cohomology of the abelian group Π (Mac Lane [1950]). This cohomology—and that of other algebraic systems—can be calculated systematically from a complex which is "generically acyclic" in the sense of Eilenberg-Mac Lane [1951] [1955]. The full meaning of this notion is still mysterious.

Subsequently these cohomology theories were unified and organized in a striking fashion by the notion of triple cohomology. This idea was an outgrowth of the notion of a pair of adjoint functors $F : X \to A$ and $U : A \to X$. Eilenberg and Moore observed that the composite endofunctor $T = UF : X \to X$ inherited from the given adjunction not only the "universal" natural transformation $\eta : I \to T$ but also a natural transformation $\mu : T^2 \to T$, with formal properties parallel to those of the multiplication μ and the unit η of a monoid or of a ring. The quadruple $< X, T, \eta, \mu >$ with these properties they called a triple, and they constructed the category of "algebras" for such a triple (better monad), to match exactly the actions of a monoid or the modules over a ring. Soon afterwards, Barr and Beck observed [1966] [1969] that these monads and these algebras could be used to systematically construct the cohomology of groups, modules, algebras and other algebraic systems. The resulting "triple cohomology" or "cotriple cohomology" was beautifully developed in an extensive seminar at the Forschungs Institut of the E.T.H. at Zurich. This development (recorded in part in a Springer Lecture Notes Vol. 80) in particular finally accounted systematically for the central role of the bar construction in all these cohomologies—thus bringing to full understanding exactly the construction first used by Eilenberg-Mac Lane to introduce the cohomology of groups. Eckmann's timely encouragement of this triple cohomology development at Zurich is another one of his major contributions to mathematics.

16. Some Historical Questions.

Our discussion has traced some of the ramifications of the development of the cohomology of groups. Inevitably it raises for consideration a number of speculative questions—which can hardly be settled by reference to this one sample piece of the history of recent mathematics.

First, a mathematical idea looks very different coming and going. The cohomology of groups started as a particular question as to a construction of part of the 2-dimensional homology groups. It also may have started as a construction to realize explicitly the meaning of that theorem of Hurewicz asserting that the fundamental group of an aspherical space determines all the homology groups. Thus the cohomology of groups, intended to provide the solution to a problem, became a theory and also became a connection (or, the discovery of a connection) between algebra and topology. This discovery came (by chance or by direct influence) at