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could be embedded in an injective (i.e., divisible) abelian group. In 1940 R. Baer, using transfinite induction, proved that the same held for R-modules over every ring. This was exactly the result necessary to construct an injective resolution for any R-module.

In 1953, Eckmann and Schopf provided a new and much more perspicuous proof that every R-module A could be embedded in an injective one. They first embedded A, regarded as an abelian group, into a divisible group D and then formed the double embedding

$$A \rightarrowtail \hom(R, A) \rightarrowtail \hom(R, D)$$

proving that D divisible meant that the $\hom(R, D)$ is injective. Going beyond this, they observed that there was in fact a *minimal* way of embedding A into an injective module J. Finding this depended on the notion of an essential extension. A submodule $A \subset B$ or a monomorphism $A \rightarrowtail B$ is *essential* if for each submodule S of B, $S \cap A = 0$ implies $S = 0$; in other words $B \supset A$ is essential if every non-trivial submodule of B must actually meet A in some non-zero elements. From this definition it is not hard to see that each module A has a *maximal* essential extension $A \rightarrowtail E$. This maximal essential extension now turns out to be the minimal injective extension of A—a result of great beauty and use.

13. Functors and Categories

In another direction, the development of the cohomology of groups was an essential preliminary to the formulation of the notions of category and functor. Hopf's discovery of the second homotopy group $H_2(G, \mathbf{Z})$ provided a highly non-trivial example of a functor of G. To be sure, this functor had been present before; in the form

$$H_2(G, \mathbf{Z}) = R \cap [F, F] / [F, F] \qquad G = F/R\,,$$

it was in fact identical with Schur's "multiplicator"—though any general description of "functors" would have been unlikely at the time when Schur was using his multiplicator in connection with projective representations. However, in 1942 the mathematical atmosphere was different and more ready for abstractions (thanks to the influence of Hilbert, Emmy Noether, and others). Moreover, there were other prominent examples of non-trivial constructions on groups which were functors—the group $\operatorname{Ext}(G, A)$ of all abelian extensions of the abelian group A by G being one. Indeed, it was principally this functor (as it was needed for the universal coefficient theorem in cohomology) that led Eilenberg-Mac Lane in 1943 to the step of introducing categories in general and functors on them, both covariant and contravariant.

The categorical language was soon generally used for homology theory and homological algebra—but one essential element of that language was missing: The notion of adjoint functor. This notion did not actually appear till D. M. Kan's clear introduction in 1958. To be sure, many special examples, usually under the form of a suitable universal property, had been long present. However, the great merit of the notion lies in its generality and systematic availability. In retrospect (see Mac Lane [1976]) it is strange indeed that it took 15 years from the introduction of categories in 1943 to the definition of adjoint functors in 1958. It may indeed be that there was a widespread prejudice against very general notions ("general abstract nonsense") and that the mores of mathematical research were determined more by a sort of positivistic view—all that matters are hard calculations leading to explicit theorems solving known problems. This clearly useful and effective standard—for most mathematical purposes—may have needlessly inhibited the development of appropriate general concepts. This is hard to judge with certainty. I do know that Eilenberg-Mac Lane for a dozen years after their initial publication on category theory considered that category theory was chiefly a language, and that further serious research in the subject was not worth trying. When Daniel Kan, coming from outside the main communities of mathematics, did arrive at the notion of a pair of adjoint functors, his work was warmly greeted by Eilenberg.

This may leave us to wonder if there are other general notions not yet discovered which might be useful for the organization of mathematics.

14. Duality

One general notion, that of categorical duality and its topological application, did not lack for attention. Pontryagin duality for topological groups had long (since about 1930) been a central tool for the algebraic topologists, especially for its use with the coefficient groups of knowledge and cohomology. The alternative possibility of dualities which are axiomatic (because they arise from a dual involution of the undefined terms of an axiom system) could not very well become relevant for topology until the categorical language was available. Possibly the first step in this direc-