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11. Class Field Theory

Some of the origins of the cohomology of groups—specifically, the factor sets for crossed product algebras—came from class field theory. Hence it is not surprising that one of the principle uses of this cohomology lies back in class field theory. Possibilities of this sort were in the minds of Eilenberg and Mac Lane when they wrote a paper applying cohomology of groups along the lines of the earlier Teichmüller work [1940] on 3-cocycles. Mac Lane also recalls that Artin (about 1948) pointed out in conversations that the cohomology of groups should have use in class field theory. Hochschild [1950] and Hochschild and Nakayama [1952] showed how the Brauer group arguments of class field theory could be replaced by cohomological arguments. In 1952, Tate proved that the homology and cohomology groups for a finite group G could be suitably combined in a single long exact sequence. He used this sequence, together with properties of transfer and restriction, to give an elegant reformulation of class field theory. It is still today one of the effective approaches to this subject—as presented, for example, in the recent book of Iyanaga and Iyanaga [1975].

12. Homological Algebra

The discovery of the cohomology of groups was an essential part of the development of homological algebra. This subject, as organized by Cartan and Eilenberg, provides a unified way of accounting for a variety of new functors, starting with the cohomology of groups. Such are:

- $H^n(G, A)$, the cohomology of a group G, with coefficients in a left G-module A;
- $H_n(G, A)$, the homology of a group G, with coefficients in a right G module A;
- $H^n(\Lambda, A)$, the (Hochschild) cohomology of an algebra Λ, with coefficients in a Λ-bimodule A;
- $H^n(g, C)$, the cohomology of the Lie algebra g, with coefficient in a g-module C;
- Ext (A, B), the group of abelian group extensions of the abelian group B by the abelian group A;
- Tor (A, B), the torsion product of the abelian groups A and B.

The first three functors (and others like them) all arose from our immediate subject, the cohomology of groups. The functor Ext is related since it describes a group of group extensions, but it enters our story more directly by its role in the universal coefficient theorem for homology; as found by Eilenberg-Mac Lane on the basis of a problem of Steenrod about regular cycles in metric spaces. Finally the Tor functor also came from the universal coefficient theorem in homology—and the functor Tor (without the name) first appears in connection with the universal coefficient theorem in a 1935 paper by Čech.

The decisive idea of homological algebra was the recognition that all these functors—as well as the higher Ext^n and $\mathrm{Tor}_n(A, B)$ for modules A and B over a ring R—could be described uniformly as the n^{th} "derived" functors of certain basic functors. Here the definition of derived functor rests on the notion of a projective resolution, which comes directly from the ideas of Hopf and Freudenthal on the homology of a group. For example, in this case, one regards the additive group $\mathbf{Z}$ of integers as a trivial $\mathbf{Z}$-module, forms an exact sequence.

$$\mathbf{Z} \leftarrow X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \ldots$$

of projective left G-modules, tensors the result with A

$$A \otimes_G X_0 \leftarrow A \otimes_G X_1 \leftarrow A \otimes_G X_2 \leftarrow \ldots$$

and calculates the homology of this complex in dimension n to obtain $H_n(G, A)$. For these homology groups, this is exactly the procedure used in Hopf's second paper to describe the Betti groups which belong to the group G—except that, as already noted, he did not have the tensor product of G modules at hand. He used only the trivial G module $A = \mathbf{Z}$, so he could describe our tensor product $\mathbf{Z} \otimes_G X$ as the quotient X/X_0, where X_0 is the submodule of X generated by all the finite sums $\Sigma\beta_i x_i$ with x_i in X, β_i in the group ring $\mathbf{Z}(G)$ and with augmentation $\alpha(\Sigma\beta_i) = 0$. In exact sequence terminology, this amounted to using the augmentation α to form a short exact sequence

$$I(G) \rightarrowtail \mathbf{Z}(G) \xrightarrow{\alpha} \mathbf{Z},$$

forming from this the right exact sequence

$$I(G) \otimes_G X \to \mathbf{Z}(G) \otimes_G X \cong X \to Z \otimes_G X \to 0$$

and hence getting $\mathbf{Z} \otimes_G X$ as the stated quotient of X. For us, it is easier now to use $\otimes_G$ directly—but Hopf's treatment shows that the ideas could still work without this explicit concept.

As observed, the strength of homological algebra lay in using the *same* method of resolution to describe derived functors of arbitrary additive functors—and this use of resolutions, together with the comparison theorem for different resolutions, came straight from the geometric properties of covering spaces as used by Hopf in his original construction. The other surprising aspect was the fact that homological algebra, formulated in this generality, had extensive applications to ring theory, especially through the consideration of homological dimension. It turned out that resolutions had really appeared before: In Hilbert's proof of his theorem on syzygies!

Actually, the complete theory of derived functors depends on the use of both projective and injective resolutions. A module P over the ring R is *projective* if every morphism f from P to the codomain of an epimorphism can be lifted to the domain as in the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "f'"'] \arrow[d, "f"] \\ A \arrow[r, two heads, "e"] & B \end{tikzcd}$$

in other words if B is the codomain of an epimorphism e, each $f: P \to B$ factors as $f = ef'$ for some f'. In particular, a free R-module is evidently projective, so there are plenty of projective modules; in particular, every module is a quotient of a projective module.

The dual notion is that of an injective module J. A left R-module J is injective if every morphism f from the domain of a monomorphism can be extended to the codomain; that is, if for each monomorphism $m: A \to B$, any $f: A \to J$ extends to an $f': B \to J$ with $f'm = f$, as in the commutative diagram

$$\begin{tikzcd} A \arrow[r, tail, "m"] \arrow[d, "f"'] & B \arrow[dl, dashed, "f'"] \\ J & \end{tikzcd}$$

In this case the existence of injectives is not so evident, except in the case of abelian groups (Z-modules) where the injectives are exactly the divisible abelian groups. However in this case it was known that every abelian group could be embedded in an injective (i.e., divisible) abelian group. In 1940 R. Baer, using transfinite induction, proved that the same held for R-modules over every ring. This was exactly the result necessary to construct an injective resolution for any R-module.

In 1953, Eckmann and Schopf provided a new and much more perspicuous proof that every R-module A could be embedded in an injective one. They first embedded A, regarded as an abelian group, into a divisible group D and then formed the double embedding

$$A \rightarrowtail \hom(R, A) \rightarrowtail \hom(R, D)$$

proving that D divisible meant that the $\hom(R, D)$ is injective. Going beyond this, they observed that there was in fact a *minimal* way of embedding A into an injective module J. Finding this depended on the notion of an essential extension. A submodule $A \subset B$ or a monomorphism $A \rightarrowtail B$ is *essential* if for each submodule S of B, $S \cap A = 0$ implies $S = 0$; in other words $B \supset A$ is essential if every non-trivial submodule of B must actually meet A in some non-zero elements. From this definition it is not hard to see that each module A has a *maximal* essential extension $A \rightarrowtail E$. This maximal essential extension now turns out to be the minimal injective extension of A—a result of great beauty and use.

13. Functors and Categories

In another direction, the development of the cohomology of groups was an essential preliminary to the formulation of the notions of category and functor. Hopf's discovery of the second homotopy group $H_2(G, \mathbf{Z})$ provided a highly non-trivial example of a functor of G. To be sure, this functor had been present before; in the form

$$H_2(G, \mathbf{Z}) = R \cap [F, F] / [F, F] \qquad G = F/R,$$

it was in fact identical with Schur's "multiplicator"—though any general description of "functors" would have been unlikely at the time when Schur was using his multiplicator in connection with projective representations. However, in 1942 the mathematical atmosphere was different and more ready for abstractions (thanks to the influence of Hilbert, Emmy Noether, and others). Moreover, there were other prominent examples of non-trivial constructions on groups which were functors—the group $\mathrm{Ext}(G, A)$ of all abelian extensions of the abelian group A by G being one. Indeed, it was