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pretation for higher n, long sought for, is still missing (and may even not be there!). Eckmann introduced G-finite cohomology groups (1947) and showed their connection with the Hopf-Freudenthal theory of the *ends* of a group. Eckmann's work, and the paper of Eilenberg-Mac Lane on complexes with operators, again emphasized the connection of cohomology groups of groups with covering spaces. There was a systematic presentation of the subject in the Cartan seminar of 1950/51, entitled "Cohomologie des groupes, suite spectrale, faisceaux". In this seminar Eilenberg first described the cohomology groups axiomatically, and then proved their existence. Subsequent exposés by Cartan emphasized the calculation of the cohomology by free resolutions complete with an abstract version of the comparison theorem. A decisive example of the effective use of such resolutions is the calculation of the cohomology of a cyclic group—carried out here in exposé 3. (I am sensitive to the advantage of using resolutions for this purpose, because in 1948 I had calculated the cohomology of cyclic groups directly from the bar resolution *without* the general comparison theorem—the direct method worked but was much more cumbersome.) Subsequent exposés made a number of applications—to the Brauer group, the Wedderburn theorem, the theorem of Maschke on complete reducibility of linear representations of a finite group, and P. A. Smith's theorem.

Further applications to pure group theory have been limited. One small but striking one is the homology proof by Gaschutz [1966]:

THEOREM. A finite non-abelian p-group has an automorphism of p^{th} power order which is not an inner automorphism.

This conference in Zurich has exhibited more examples of the use of homology in group theory.

9. SPECTRAL SEQUENCES

The results stimulated by group cohomology were not confined just to group theory. For example, the problem of computing the cohomology groups $H^n(G, A)$ for the case when G itself is a group extension (say, cyclic by cyclic) immediately leads to the study of a spectral sequence. Specifically, if

$$1 \to K \to G \to Q \to 1 \qquad (1)$$

is a short exact sequence of (multiplicative) groups and A is a left G module there is a spectral sequence E_r^{pq} with

$$E_2^{pq} \cong H^p(Q, H^q(K, M)) \tag{2}$$

converging to the graded group associated with a filtration of the cohomology $E^{p+q}(G, M)$. In (2), the cohomology $H^q(K, M)$ of the subgroup K is suitably interpreted as a Q-module, so that the outside cohomology is defined. The essential portions of such a spectral sequence were discovered by R. Lyndon in his 1946 Harvard thesis, at about the same time that Leray was formulating the general notion of a spectral sequence. Lyndon did use his formulation for computation. Some years later [1953], Hochschild and Serre formulated a spectral sequence like that of (2) in the conventional language, so such a sequence is usually called a Hochschild-Serre spectral sequence. (There are actually several different constructions of such a sequence, and some residual uncertainty as to whether these constructions all yield the same spectral sequence). The essential observation is that computing cohomology or homology in a fiber situation like that of (1) inevitably leads to the spectral sequence technology—whether the fiber situation is group theoretic, as with the exact sequence (1), or a fiber space, as in the case so effectively exploited by Serre in topology.

10. Transfer

The operation of *transfer* was well known in group theory, beginning with Burnside's work on monomial representations. If H is a subgroup of index n in G, the transfer from G to H is a homomorphism.

$$t : G/[G, G] \to H/[H, H] \tag{1}$$

between the factor-commutator groups. To define it, choose representatives $x_1, \ldots, x_n$ of the right cosets of H in G, so that $G = \cup Hx_i$ and write $\rho(x)$ for the representative x_i of the coset Hx. Then t is

$$t(g) = \prod_{i=1}^{n} (x_i g)\,[\rho(x_i g)]^{-1} \tag{2}$$

This map t is independent of the choice of the set of representatives $x_i, \ldots, x_n$.

Since the factor commutator group $G/[G, G]$ in (1) is simply the 1-dimensional homology group $H_1(G, \mathbf{Z})$, the transfer can be regarded as a map in homology.

$$t : H_1(G, \mathbf{Z}) \to H_1(H, \mathbf{Z})$$

In 1953 Eckmann extended this map to apply in all dimensions, both in homology and cohomology. Using the standard homogeneous complexes