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Artikel: SIMPLE PROOF OF THE MAIN THEOREM OF ELIMINATION THEORY IN ALGEBRAIC GEOMETRY

Autor: Cartier, P. / Tate, J.

Kapitel: 4. Proof of theorem D

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(i) *As a ring,* A *is generated by* $A_0 \cup A_1$.

(ii) *For any nonnegative integer* d, A_d *is a finitely generated module over* A_0.

Furthermore, let $\mathfrak{S}$ be the ideal in A_0 consisting of all a's such that $aA_d = 0$ for all sufficiently large d's, i.e. the union of the annihilators of the A_0-modules $A_0, A_1, A_2, \ldots$.

THEOREM D. *Let* $A = \bigoplus_{d \geqq 0} A_d$ *be a graded commutative ring obeying hypotheses* (i) *and* (ii) *above. Let* K *be an algebraically closed field and* $\varphi : A_0 \to K$ *be a ring homomorphism. In order that* φ *extend to a ring homomorphism* $\Psi : A \to K$ *which does not annihilate the ideal* $A^+ = \bigoplus_{d \geqq 1} A_d$ *in* A, *it is necessary and sufficient that* φ *annihilate the ideal* $\mathfrak{S}$ *defined above.*

We leave to the reader the simple proof of the necessity in theorem D as well as the derivation of theorem C from theorem D.

4. PROOF OF THEOREM D

Let $\mathfrak{P}$ be the kernel of φ, a prime ideal in A_0. Assume $\mathfrak{S} \subset \mathfrak{P}$. We subject the ring A to a number of transformations. At each step, the properties (i) and (ii) enunciated before the statement of theorem D will be preserved, as well as property $A_d \neq 0$ for every $d \geqslant 0$. We shall mention what has been achieved after each step.

a) Factor A through the following graded ideal J: an element a in A belongs to J if and only if there exists an element s in A_0 such $s \notin \mathfrak{P}$ and $sa = 0$. For every $d \geqslant 0$, the annihilator $\mathfrak{S}_d$ of the A_0-module A_d is contained in $\mathfrak{S}$ hence in $\mathfrak{P}$ and this implies $J \cap A_d \neq A_d$. Put $A' = A/J$, $\mathfrak{P}' = (\mathfrak{P} + J)/J$ and $\Sigma = A_0' - \mathfrak{P}'$. Then *any element in* Σ *is regular in* A'.

b) Enlarge A' by replacing it by the subring A'' of the total quotient ring of A' consisting of the fractions with denominators in Σ. Let A_d'' be the set of fractions with numerator in A_d' and denominator in Σ; then $A'' = \bigoplus_{d \geqq 0} A_d''$. Then A_0'' *is a local ring with maximal ideal* $\mathfrak{P}'' = \mathfrak{P}' \,.\, A_0'$.

c) Factor A'' through the graded ideal $\mathfrak{P}'' \,.\, A''$. Since A_d'' is a finitely generated module over the local ring A_0'', one gets $A_d'' \neq \mathfrak{P}'' A_d''$ by Nakayama's lemma. Put $k = A_0'' \backslash \mathfrak{P}''$, and $R = A''/\mathfrak{P}'' A''$.

At this point, k is a field (the quotient field of $A_0/\mathfrak{P}$) and R is a graded algebra over the field k, so all assumptions of theorem B are fulfilled. Moreover let ε the composition of the natural maps

$$A \to A' \to A'' \to R\,.$$

In degree 0, ε_0 is nothing else than the natural map from A_0 into k with kernel $\mathfrak{P}$. Since φ has the same kernel $\mathfrak{P}$, it factors through ε_0, making K an algebraically closed extension of k.

We quote now theorem B. There exists a k-linear ring homomorphism $f: R \to K$ such that $f(R^+) \neq 0$. The composite map $\Psi = f\varepsilon$ has all the required properties.

5. Application to schemes

We keep the notation of theorem D. Recall that the spectrum $S = \mathbf{Spec}\,(A_0)$ of A_0 is the set of all prime ideals in A_0; the projective spectrum $X = \mathbf{Proj}\,(A)$ of A is the set of all *graded* prime ideals in A, which do not contain the ideal $A^+ = \bigoplus_{d \geqq 1} A_d$. We have a natural map $\pi : X \to S$ associating to every graded prime ideal $\mathfrak{P}$ in A the prime ideal $\mathfrak{P} \cap A_0$ in A_0.

Moreover S and X are endowed with their respective Zariski topologies. A set F in S (resp. X) is closed if and only if there exists an ideal $\mathfrak{A}$ in A_0 (resp. A) such that F is the set of ideals $\mathfrak{P}$ of S (resp. X) containing $\mathfrak{A}$. It is obvious that π is continuous.

The following theorem is Grothendieck's version of the elimination theorem. Using his language, it is the main step in the proof that $X = \mathbf{Proj}\,(A)$ is a proper scheme over $S = \mathbf{Spec}\,(A_0)$.

Theorem E. *The map $\pi : X \to S$ is closed, that is the image of a closed set is closed.*

Let $F \subset X$ be closed and let $\mathfrak{A}$ be an ideal in A such that F consists of the graded prime ideals $\mathfrak{P}$ of X containing $\mathfrak{A}$. Replacing if necessary $\mathfrak{A}$ by the ideal generated by the homogeneous components of its elements, we may and shall assume that $\mathfrak{A}$ is a graded ideal. Let $\mathfrak{B}$ be the set of elements a in A_0 such that $a\,.\,A_d \subset \mathfrak{A}$ for large d, and let G be the set of prime ideals in A_0 containing $\mathfrak{B}$. It is obvious that π maps F into G.

Let $\mathfrak{P}_0$ be a prime ideal in G, hence $\mathfrak{P}_0 \supset \mathfrak{A}_0$ (where $\mathfrak{A}_0 = \mathfrak{A} \cap A_0$). Denote by k the quotient field of $A_0/\mathfrak{P}_0$ and by K an algebraically closed