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- (i) As a ring,  $A$  is generated by  $A_0 \cup A_1$ .
- (ii) For any nonnegative integer  $d$ ,  $A_d$  is a finitely generated module over  $A_0$ .

Furthermore, let  $\mathfrak{S}$  be the ideal in  $A_0$  consisting of all  $a$ 's such that  $aA_d = 0$  for all sufficiently large  $d$ 's, i.e. the union of the annihilators of the  $A_0$ -modules  $A_0, A_1, A_2, \dots$ .

**THEOREM D.** Let  $A = \bigoplus_{d \geq 0} A_d$  be a graded commutative ring obeying hypotheses (i) and (ii) above. Let  $K$  be an algebraically closed field and  $\varphi : A_0 \rightarrow K$  be a ring homomorphism. In order that  $\varphi$  extend to a ring homomorphism  $\Psi : A \rightarrow K$  which does not annihilate the ideal  $A^+ = \bigoplus_{d \geq 1} A_d$  in  $A$ , it is necessary and sufficient that  $\varphi$  annihilate the ideal  $\mathfrak{S}$  defined above.

We leave to the reader the simple proof of the necessity in theorem D as well as the derivation of theorem C from theorem D.

#### 4. PROOF OF THEOREM D

Let  $\mathfrak{P}$  be the kernel of  $\varphi$ , a prime ideal in  $A_0$ . Assume  $\mathfrak{S} \subset \mathfrak{P}$ . We subject the ring  $A$  to a number of transformations. At each step, the properties (i) and (ii) enunciated before the statement of theorem D will be preserved, as well as property  $A_d \neq 0$  for every  $d \geq 0$ . We shall mention what has been achieved after each step.

a) Factor  $A$  through the following graded ideal  $J$ : an element  $a$  in  $A$  belongs to  $J$  if and only if there exists an element  $s$  in  $A_0$  such  $s \notin \mathfrak{P}$  and  $sa = 0$ . For every  $d \geq 0$ , the annihilator  $\mathfrak{S}_d$  of the  $A_0$ -module  $A_d$  is contained in  $\mathfrak{S}$  hence in  $\mathfrak{P}$  and this implies  $J \cap A_d \neq A_d$ . Put  $A' = A/J$ ,  $\mathfrak{P}' = (\mathfrak{P} + J)/J$  and  $\Sigma = A'_0 - \mathfrak{P}'$ . Then any element in  $\Sigma$  is regular in  $A'$ .

b) Enlarge  $A'$  by replacing it by the subring  $A''$  of the total quotient ring of  $A'$  consisting of the fractions with denominators in  $\Sigma$ . Let  $A''_d$  be the set of fractions with numerator in  $A'_d$  and denominator in  $\Sigma$ ; then  $A'' = \bigoplus_{d \geq 0} A''_d$ . Then  $A''_0$  is a local ring with maximal ideal  $\mathfrak{P}'' = \mathfrak{P}' \cdot A'_0$ .

c) Factor  $A''$  through the graded ideal  $\mathfrak{P}'' \cdot A''$ . Since  $A''_d$  is a finitely generated module over the local ring  $A''_0$ , one gets  $A''_d \neq \mathfrak{P}'' A''_d$  by Nakayama's lemma. Put  $k = A''_0 / \mathfrak{P}''$ , and  $R = A'' / \mathfrak{P}'' A''$ .

At this point,  $k$  is a field (the quotient field of  $A_0/\mathfrak{P}$ ) and  $R$  is a graded algebra over the field  $k$ , so all assumptions of theorem B are fulfilled. Moreover let  $\varepsilon$  the composition of the natural maps

$$A \rightarrow A' \rightarrow A'' \rightarrow R.$$

In degree 0,  $\varepsilon_0$  is nothing else than the natural map from  $A_0$  into  $k$  with kernel  $\mathfrak{P}$ . Since  $\varphi$  has the same kernel  $\mathfrak{P}$ , it factors through  $\varepsilon_0$ , making  $K$  an algebraically closed extension of  $k$ .

We quote now theorem B. There exists a  $k$ -linear ring homomorphism  $f: R \rightarrow K$  such that  $f(R^+) \neq 0$ . The composite map  $\Psi = f\varepsilon$  has all the required properties.

## 5. APPLICATION TO SCHEMES

We keep the notation of theorem D. Recall that the spectrum  $S = \mathbf{Spec}(A_0)$  of  $A_0$  is the set of all prime ideals in  $A_0$ ; the projective spectrum  $X = \mathbf{Proj}(A)$  of  $A$  is the set of all *graded* prime ideals in  $A$ , which do not contain the ideal  $A^+ = \bigoplus_{d \geq 1} A_d$ . We have a natural map  $\pi: X \rightarrow S$  associating to every graded prime ideal  $\mathfrak{P}$  in  $A$  the prime ideal  $\mathfrak{P} \cap A_0$  in  $A_0$ .

Moreover  $S$  and  $X$  are endowed with their respective Zariski topologies. A set  $F$  in  $S$  (resp.  $X$ ) is closed if and only if there exists an ideal  $\mathfrak{U}$  in  $A_0$  (resp.  $A$ ) such that  $F$  is the set of ideals  $\mathfrak{P}$  of  $S$  (resp.  $X$ ) containing  $\mathfrak{U}$ . It is obvious that  $\pi$  is continuous.

The following theorem is Grothendieck's version of the elimination theorem. Using his language, it is the main step in the proof that  $X = \mathbf{Proj}(A)$  is a proper scheme over  $S = \mathbf{Spec}(A_0)$ .

**THEOREM E.** *The map  $\pi: X \rightarrow S$  is closed, that is the image of a closed set is closed.*

Let  $F \subset X$  be closed and let  $\mathfrak{U}$  be an ideal in  $A$  such that  $F$  consists of the graded prime ideals  $\mathfrak{P}$  of  $X$  containing  $\mathfrak{U}$ . Replacing if necessary  $\mathfrak{U}$  by the ideal generated by the homogeneous components of its elements, we may and shall assume that  $\mathfrak{U}$  is a graded ideal. Let  $\mathfrak{B}$  be the set of elements  $a$  in  $A_0$  such that  $a \cdot A_d \subset \mathfrak{U}$  for large  $d$ , and let  $G$  be the set of prime ideals in  $A_0$  containing  $\mathfrak{B}$ . It is obvious that  $\pi$  maps  $F$  into  $G$ .

Let  $\mathfrak{P}_0$  be a prime ideal in  $G$ , hence  $\mathfrak{P}_0 \supset \mathfrak{U}_0$  (where  $\mathfrak{U}_0 = \mathfrak{U} \cap A_0$ ). Denote by  $k$  the quotient field of  $A_0/\mathfrak{P}_0$  and by  $K$  an algebraically closed