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Artikel: SIMPLE PROOF OF THE MAIN THEOREM OF ELIMINATION THEORY IN ALGEBRAIC GEOMETRY

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$I_0 = (0)$ and $\mathfrak{S}_0 = \mathfrak{S}$ and define inductively r_d, I_d and \mathfrak{S}_d as follows. For $d \geq 0$, let r_{d+1} be equal to the maximum of the dimensions of $I \cap R_{d+1}$ for I running over \mathfrak{S}_d , let I_{d+1} be any ideal in \mathfrak{S}_d such that $\dim(I_{d+1} \cap R_{d+1}) = r_{d+1}$ and let \mathfrak{S}_{d+1} be the set of ideals I in \mathfrak{S}_d such that $I \cap R_{d+1} = I_{d+1} \cap R_{d+1}$. Then the ideal $\bigoplus_{d \geq 1} (I_d \cap R_d)$ is a maximal element in \mathfrak{S} , as it is easily checked.

3. ELIMINATION THEORY

The main theorem of elimination theory may be formulated as follows. Let P_1, \dots, P_r be polynomials in $k[X_0, X_1, \dots, X_n; Y_1, \dots, Y_m]$ with P_j homogeneous of degree d_j in the variables X_0, X_1, \dots, X_n alone, i.e. of the form

$$P_j = \sum_{\alpha_0 + \dots + \alpha_n = d_j} X_0^{\alpha_0} X_1^{\alpha_1} \dots X_n^{\alpha_n} f_{\alpha,j}(Y_1, \dots, Y_m)$$

where the $f_{\alpha,j}$'s are polynomials in $k[Y_1, \dots, Y_m]$.

Denote by J the ideal in $k[X_0, X_1, \dots, X_n; Y_1, \dots, Y_m]$ generated by P_1, \dots, P_r and by \mathfrak{A} the ideal of polynomials f in $k[Y_1, \dots, Y_m]$ with the following property (the so-called Hurwitz' Trägheitsformen):

(E) There exists an integer $N \geq 1$ such that $f X_0^N, f X_1^N, \dots, f X_n^N$ all belong to J .

As usual we denote by $\mathbf{P}^n(K)$ the n -dimensional projective space over K .

THEOREM C. *Let V be the subset of $\mathbf{P}^n(K) \times K^m$ consisting of the pairs (x, y) with $x = (x_0 : x_1 : \dots : x_n)$ and $y = (y_1, \dots, y_m)$ such that $P_j(x_0, x_1, \dots, x_n; y_1, \dots, y_m) = 0$ for $1 \leq j \leq r$. Let W be the subset of K^m consisting of the vectors y such that $Q(y) = 0$ for every Q in \mathfrak{A} . Then the projection of $V \subset \mathbf{P}^n(K) \times K^m$ onto the second factor K^m is equal to W .*

To reformulate theorem C, let us consider the ring

$$B = k[X_0, X_1, \dots, X_n; Y_1, \dots, Y_m]$$

together with its subring $B_0 = k[Y_1, \dots, Y_m]$. Denote by B_d the B_0 -module generated in B by the monomials of degree d in X_0, X_1, \dots, X_n . Then $B = \bigoplus_{d \geq 0} B_d$ is a graded ring with J a graded ideal. Define the *graded ring* $A = B/J$ with $A_d = B_d/(B_d \cap J)$. We have the following properties:

- (i) As a ring, A is generated by $A_0 \cup A_1$.
- (ii) For any nonnegative integer d , A_d is a finitely generated module over A_0 .

Furthermore, let \mathfrak{S} be the ideal in A_0 consisting of all a 's such that $aA_d = 0$ for all sufficiently large d 's, i.e. the union of the annihilators of the A_0 -modules A_0, A_1, A_2, \dots .

THEOREM D. *Let $A = \bigoplus_{d \geq 0} A_d$ be a graded commutative ring obeying hypotheses (i) and (ii) above. Let K be an algebraically closed field and $\varphi : A_0 \rightarrow K$ be a ring homomorphism. In order that φ extend to a ring homomorphism $\Psi : A \rightarrow K$ which does not annihilate the ideal $A^+ = \bigoplus_{d \geq 1} A_d$ in A , it is necessary and sufficient that φ annihilate the ideal \mathfrak{S} defined above.*

We leave to the reader the simple proof of the necessity in theorem D as well as the derivation of theorem C from theorem D.

4. PROOF OF THEOREM D

Let \mathfrak{P} be the kernel of φ , a prime ideal in A_0 . Assume $\mathfrak{S} \subset \mathfrak{P}$. We subject the ring A to a number of transformations. At each step, the properties (i) and (ii) enunciated before the statement of theorem D will be preserved, as well as property $A_d \neq 0$ for every $d \geq 0$. We shall mention what has been achieved after each step.

a) Factor A through the following graded ideal J : an element a in A belongs to J if and only if there exists an element s in A_0 such $s \notin \mathfrak{P}$ and $sa = 0$. For every $d \geq 0$, the annihilator \mathfrak{S}_d of the A_0 -module A_d is contained in \mathfrak{S} hence in \mathfrak{P} and this implies $J \cap A_d \neq A_d$. Put $A' = A/J$, $\mathfrak{P}' = (\mathfrak{P} + J)/J$ and $\Sigma = A'_0 - \mathfrak{P}'$. Then any element in Σ is regular in A' .

b) Enlarge A' by replacing it by the subring A'' of the total quotient ring of A' consisting of the fractions with denominators in Σ . Let A''_d be the set of fractions with numerator in A'_d and denominator in Σ ; then $A'' = \bigoplus_{d \geq 0} A''_d$. Then A''_0 is a local ring with maximal ideal $\mathfrak{P}'' = \mathfrak{P}' \cdot A'_0$.

c) Factor A'' through the graded ideal $\mathfrak{P}'' \cdot A''$. Since A''_d is a finitely generated module over the local ring A''_0 , one gets $A''_d \neq \mathfrak{P}'' A''_d$ by Nakayama's lemma. Put $k = A''_0 / \mathfrak{P}''$, and $R = A'' / \mathfrak{P}'' A''$.