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$S = \bigoplus_{d \geq 0} S_d$ , and for the multiplication one gets  $S_d \cdot S_e \subset S_{d+e}$ . Otherwise stated,  $S$  is a graded algebra over the field  $k$ . Since  $J$  is generated by homogeneous polynomials, it is a graded ideal, namely  $J = \bigoplus_{d \geq 0} (J \cap S_d)$ .

The factor algebra  $R = S/J$  is therefore graded with  $R_d = S_d/(J \cap S_d)$  for any nonnegative integer  $d$ . It enjoys the following properties:

- (i) As a ring,  $R$  is generated by  $R_0 \cup R_1$ .
- (ii) For any nonnegative integer  $d$ , the vector space  $R_d$  is finite-dimensional over  $k$ .
- (iii)  $R_0 = k$ .

Denote by  $x_0, x_1, \dots, x_n$  respectively the cosets of  $X_0, X_1, \dots, X_n$  modulo  $J$ . Let  $\varphi$  be any  $k$ -linear ring homomorphism from  $R$  into  $K$ , and put  $\xi_0 = \varphi(x_0), \dots, \xi_n = \varphi(x_n)$ . It is clear that the vector  $\xi = (\xi_0, \xi_1, \dots, \xi_n)$  is a common zero of the polynomials in  $J$ . Conversely, for any such common zero, there exists a unique  $k$ -linear ring homomorphism  $\varphi : R \rightarrow K$  such that  $\xi_0 = \varphi(x_0), \dots, \xi_n = \varphi(x_n)$ . The vector  $\xi$  is equal to zero if and only if  $\varphi$  maps  $R_1 = kx_0 + \dots + kx_n$  onto 0, that is if and only if the kernel of  $\varphi$  is equal to the ideal  $R^+ = \bigoplus_{d \geq 0} R_d$  in  $R$ .

Theorem A is therefore equivalent to the following.

**THEOREM B.** *Let  $R$  be a graded commutative algebra over  $k$ , satisfying hypotheses (i), (ii) and (iii) above. One has the following dichotomy:*

- a) *Either there exists a non-negative integer  $d_0$  such that  $R_d = 0$  for  $d \geq d_0$ ;*
- b) *or for every nonnegative integer  $d$ , one has  $R_d \neq 0$  and there exists a  $k$ -linear ring homomorphism  $\varphi : R \rightarrow K$  whose kernel is different from  $R^+ = \bigoplus_{d \geq 1} R_d$ .*

Notice that  $R$  is a finite-dimensional vector space in case a), infinite-dimensional in case b).

## 2. PROOF OF HILBERT'S ZERO THEOREM

We proceed to the proof of theorem B.

By property (i) above, one gets  $R_1 \cdot R_d = R_{d+1}$  hence  $R_d = 0$  implies  $R_{d+1} = 0$ . Hence either  $R_d$  is 0 for all sufficiently large  $d$ 's, or  $R_d \neq 0$

for every  $d$ . From now on, assume we are in the second case. Since  $R$  is generated over the field  $k$  by a finite number of elements, the maximum condition holds for the ideals in  $R$ . We can therefore select a maximal element in the set  $\mathfrak{S}$  of graded ideals  $I$  in  $R$  such that  $R_d \neq I \cap R_d$  for every nonnegative integer  $d$  (notice  $(0)$  belongs to  $\mathfrak{S}$ , hence  $\mathfrak{S}$  is nonempty). Replacing  $R$  by  $R/I$ , we may assume that  $R$  enjoys the following property:

(M) *For every nonnegative integer  $d$ , one has  $R_d \neq 0$ . Every graded ideal  $I \neq (0)$  in  $R$  contains  $R_d$  for all sufficiently large  $d$ 's.*

We claim that  $R_1$  contains a non-nilpotent element. Assume the converse and let  $a_1, \dots, a_r$  be a linear basis of  $R_1$  over  $k$ . There would then exist an integer  $N \geq 1$  such that  $a_1^N = \dots = a_r^N = 0$ , any monomial of degree  $> Nr$  in  $a_1, \dots, a_r$  would be equal to zero, and we would have  $R_d = 0$  for any integer  $d > Nr$ , contrary to assumption (M).

Pick a non-nilpotent element  $x$  in  $R_1$ . The element  $1 - x$  has no inverse in  $R$ . Indeed  $x^d$  belongs to  $R_d$  for any  $d \geq 0$ , and the inverse to  $1 - x$  would be congruent to  $1 + x + x^2 + \dots + x^d$  modulo the ideal  $\sum_{i>d} R_i$  for every

$d \geq 1$ , contrary to the assumption that  $R$  is the direct sum of the  $R_d$ 's. By Krull's theorem, we may select a maximal ideal  $M$  in  $R$  containing  $1 - x$ . Then  $L = R/M$  is a field extension of  $k$ , and the element  $x$  of  $R_1$  satisfies  $x \equiv 1 \pmod{M}$ . Since  $K$  is an algebraically closed extension of  $k$ , it remains to show that  $L$  is of finite degree over  $k$ , hence isomorphic to a subextension of  $K$ .

Since  $x \cdot R = \bigoplus_{d \geq 0} x \cdot R_d$  is a graded ideal in  $R$ , one gets from (M) the existence of an integer  $d_0 \geq 0$  such that  $x \cdot R_d = R_{d+1}$  for  $d \geq d_0$ . Hence, as a module over its subring  $k[x]$ ,  $R$  is generated by  $R_0 + R_1 + \dots + R_{d_0}$  hence by a (finite) basis  $b_1, \dots, b_N$  of this vector space over  $k$ . That is, any element  $u$  in  $R$  is of the form

$$(1) \quad u = b_1 f_1(x) + \dots + b_N f_N(x)$$

where  $f_1, \dots, f_N$  are polynomials in one indeterminate with coefficients in  $k$ . From (1) one gets

$$u \equiv b_1 f_1(1) + \dots + b_N f_N(1) \pmod{M},$$

hence  $[L : k] \leq N$  is finite.

Q.E.D.

For the reader who doesn't want to appeal to Hilbert's basis theorem, here is a direct construction of a maximal element in  $\mathfrak{S}$ . Let  $r_0 = 0$ ,

$I_0 = (0)$  and  $\mathfrak{S}_0 = \mathfrak{S}$  and define inductively  $r_d, I_d$  and  $\mathfrak{S}_d$  as follows. For  $d \geq 0$ , let  $r_{d+1}$  be equal to the maximum of the dimensions of  $I \cap R_{d+1}$  for  $I$  running over  $\mathfrak{S}_d$ , let  $I_{d+1}$  be any ideal in  $\mathfrak{S}_d$  such that  $\dim(I_{d+1} \cap R_{d+1}) = r_{d+1}$  and let  $\mathfrak{S}_{d+1}$  be the set of ideals  $I$  in  $\mathfrak{S}_d$  such that  $I \cap R_{d+1} = I_{d+1} \cap R_{d+1}$ . Then the ideal  $\bigoplus_{d \geq 1} (I_d \cap R_d)$  is a maximal element in  $\mathfrak{S}$ , as it is easily checked.

### 3. ELIMINATION THEORY

The main theorem of elimination theory may be formulated as follows. Let  $P_1, \dots, P_r$  be polynomials in  $k[X_0, X_1, \dots, X_n; Y_1, \dots, Y_m]$  with  $P_j$  homogeneous of degree  $d_j$  in the variables  $X_0, X_1, \dots, X_n$  alone, i.e. of the form

$$P_j = \sum_{\alpha_0 + \dots + \alpha_n = d_j} X_0^{\alpha_0} X_1^{\alpha_1} \dots X_n^{\alpha_n} f_{\alpha,j}(Y_1, \dots, Y_m)$$

where the  $f_{\alpha,j}$ 's are polynomials in  $k[Y_1, \dots, Y_m]$ .

Denote by  $J$  the ideal in  $k[X_0, X_1, \dots, X_n; Y_1, \dots, Y_m]$  generated by  $P_1, \dots, P_r$  and by  $\mathfrak{A}$  the ideal of polynomials  $f$  in  $k[Y_1, \dots, Y_m]$  with the following property (the so-called Hurwitz' Trägheitsformen):

(E) There exists an integer  $N \geq 1$  such that  $f X_0^N, f X_1^N, \dots, f X_n^N$  all belong to  $J$ .

As usual we denote by  $\mathbf{P}^n(K)$  the  $n$ -dimensional projective space over  $K$ .

**THEOREM C.** *Let  $V$  be the subset of  $\mathbf{P}^n(K) \times K^m$  consisting of the pairs  $(x, y)$  with  $x = (x_0 : x_1 : \dots : x_n)$  and  $y = (y_1, \dots, y_m)$  such that  $P_j(x_0, x_1, \dots, x_n; y_1, \dots, y_m) = 0$  for  $1 \leq j \leq r$ . Let  $W$  be the subset of  $K^m$  consisting of the vectors  $y$  such that  $Q(y) = 0$  for every  $Q$  in  $\mathfrak{A}$ . Then the projection of  $V \subset \mathbf{P}^n(K) \times K^m$  onto the second factor  $K^m$  is equal to  $W$ .*

To reformulate theorem C, let us consider the ring

$$B = k[X_0, X_1, \dots, X_n; Y_1, \dots, Y_m]$$

together with its subring  $B_0 = k[Y_1, \dots, Y_m]$ . Denote by  $B_d$  the  $B_0$ -module generated in  $B$  by the monomials of degree  $d$  in  $X_0, X_1, \dots, X_n$ . Then  $B = \bigoplus_{d \geq 0} B_d$  is a graded ring with  $J$  a graded ideal. Define the *graded ring*  $A = B/J$  with  $A_d = B_d/(B_d \cap J)$ . We have the following properties: