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$S = \bigoplus_{d \geq 0} S_d$, and for the multiplication one gets $S_d \cdot S_e \subset S_{d+e}$. Otherwise stated, S is a graded algebra over the field k . Since J is generated by homogeneous polynomials, it is a graded ideal, namely $J = \bigoplus_{d \geq 0} (J \cap S_d)$.

The factor algebra $R = S/J$ is therefore graded with $R_d = S_d/(J \cap S_d)$ for any nonnegative integer d . It enjoys the following properties:

- (i) As a ring, R is generated by $R_0 \cup R_1$.
- (ii) For any nonnegative integer d , the vector space R_d is finite-dimensional over k .
- (iii) $R_0 = k$.

Denote by x_0, x_1, \dots, x_n respectively the cosets of X_0, X_1, \dots, X_n modulo J . Let φ be any k -linear ring homomorphism from R into K , and put $\xi_0 = \varphi(x_0), \dots, \xi_n = \varphi(x_n)$. It is clear that the vector $\xi = (\xi_0, \xi_1, \dots, \xi_n)$ is a common zero of the polynomials in J . Conversely, for any such common zero, there exists a unique k -linear ring homomorphism $\varphi : R \rightarrow K$ such that $\xi_0 = \varphi(x_0), \dots, \xi_n = \varphi(x_n)$. The vector ξ is equal to zero if and only if φ maps $R_1 = kx_0 + \dots + kx_n$ onto 0, that is if and only if the kernel of φ is equal to the ideal $R^+ = \bigoplus_{d \geq 1} R_d$ in R .

Theorem A is therefore equivalent to the following.

THEOREM B. *Let R be a graded commutative algebra over k , satisfying hypotheses (i), (ii) and (iii) above. One has the following dichotomy:*

- a) *Either there exists a non-negative integer d_0 such that $R_d = 0$ for $d \geq d_0$;*
- b) *or for every nonnegative integer d , one has $R_d \neq 0$ and there exists a k -linear ring homomorphism $\varphi : R \rightarrow K$ whose kernel is different from $R^+ = \bigoplus_{d \geq 1} R_d$.*

Notice that R is a finite-dimensional vector space in case a), infinite-dimensional in case b).

2. PROOF OF HILBERT'S ZERO THEOREM

We proceed to the proof of theorem B.

By property (i) above, one gets $R_1 \cdot R_d = R_{d+1}$ hence $R_d = 0$ implies $R_{d+1} = 0$. Hence either R_d is 0 for all sufficiently large d 's, or $R_d \neq 0$

for every d . From now on, assume we are in the second case. Since R is generated over the field k by a finite number of elements, the maximum condition holds for the ideals in R . We can therefore select a maximal element in the set \mathfrak{S} of graded ideals I in R such that $R_d \neq I \cap R_d$ for every nonnegative integer d (notice (0) belongs to \mathfrak{S} , hence \mathfrak{S} is nonempty). Replacing R by R/I , we may assume that R enjoys the following property:

(M) *For every nonnegative integer d , one has $R_d \neq 0$. Every graded ideal $I \neq (0)$ in R contains R_d for all sufficiently large d 's.*

We claim that R_1 contains a non-nilpotent element. Assume the converse and let a_1, \dots, a_r be a linear basis of R_1 over k . There would then exist an integer $N \geq 1$ such that $a_1^N = \dots = a_r^N = 0$, any monomial of degree $> Nr$ in a_1, \dots, a_r would be equal to zero, and we would have $R_d = 0$ for any integer $d > Nr$, contrary to assumption (M).

Pick a non-nilpotent element x in R_1 . The element $1 - x$ has no inverse in R . Indeed x^d belongs to R_d for any $d \geq 0$, and the inverse to $1 - x$ would be congruent to $1 + x + x^2 + \dots + x^d$ modulo the ideal $\sum_{i>d} R_i$ for every

$d \geq 1$, contrary to the assumption that R is the direct sum of the R_d 's. By Krull's theorem, we may select a maximal ideal M in R containing $1 - x$. Then $L = R/M$ is a field extension of k , and the element x of R_1 satisfies $x \equiv 1 \pmod{M}$. Since K is an algebraically closed extension of k , it remains to show that L is of finite degree over k , hence isomorphic to a subextension of K .

Since $x \cdot R = \bigoplus_{d \geq 0} x \cdot R_d$ is a graded ideal in R , one gets from (M) the existence of an integer $d_0 \geq 0$ such that $x \cdot R_d = R_{d+1}$ for $d \geq d_0$. Hence, as a module over its subring $k[x]$, R is generated by $R_0 + R_1 + \dots + R_{d_0}$ hence by a (finite) basis b_1, \dots, b_N of this vector space over k . That is, any element u in R is of the form

$$(1) \quad u = b_1 f_1(x) + \dots + b_N f_N(x)$$

where f_1, \dots, f_N are polynomials in one indeterminate with coefficients in k . From (1) one gets

$$u \equiv b_1 f_1(1) + \dots + b_N f_N(1) \pmod{M},$$

hence $[L : k] \leq N$ is finite.

Q.E.D.

For the reader who doesn't want to appeal to Hilbert's basis theorem, here is a direct construction of a maximal element in \mathfrak{S} . Let $r_0 = 0$,

$I_0 = (0)$ and $\mathfrak{I}_0 = \mathfrak{I}$ and define inductively r_d , I_d and \mathfrak{I}_d as follows. For $d \geq 0$, let r_{d+1} be equal to the maximum of the dimensions of $I \cap R_{d+1}$ for I running over \mathfrak{I}_d , let I_{d+1} be any ideal in \mathfrak{I}_d such that $\dim(I_{d+1} \cap R_{d+1}) = r_{d+1}$ and let \mathfrak{I}_{d+1} be the set of ideals I in \mathfrak{I}_d such that $I \cap R_{d+1} = I_{d+1} \cap R_{d+1}$. Then the ideal $\bigoplus_{d \geq 1} (I_d \cap R_d)$ is a maximal element in \mathfrak{I} , as it is easily checked.

3. ELIMINATION THEORY

The main theorem of elimination theory may be formulated as follows. Let P_1, \dots, P_r be polynomials in $k[X_0, X_1, \dots, X_n; Y_1, \dots, Y_m]$ with P_j homogeneous of degree d_j in the variables X_0, X_1, \dots, X_n alone, i.e. of the form

$$P_j = \sum_{\alpha_0 + \dots + \alpha_n = d_j} X_0^{\alpha_0} X_1^{\alpha_1} \dots X_n^{\alpha_n} f_{\alpha,j}(Y_1, \dots, Y_m)$$

where the $f_{\alpha,j}$'s are polynomials in $k[Y_1, \dots, Y_m]$.

Denote by J the ideal in $k[X_0, X_1, \dots, X_n; Y_1, \dots, Y_m]$ generated by P_1, \dots, P_r and by \mathfrak{A} the ideal of polynomials f in $k[Y_1, \dots, Y_m]$ with the following property (the so-called Hurwitz' Trägheitsformen):

(E) There exists an integer $N \geq 1$ such that $fX_0^N, fX_1^N, \dots, fX_n^N$ all belong to J .

As usual we denote by $\mathbf{P}^n(K)$ the n -dimensional projective space over K .

THEOREM C. *Let V be the subset of $\mathbf{P}^n(K) \times K^m$ consisting of the pairs (x, y) with $x = (x_0 : x_1 : \dots : x_n)$ and $y = (y_1, \dots, y_m)$ such that $P_j(x_0, x_1, \dots, x_n; y_1, \dots, y_m) = 0$ for $1 \leq j \leq r$. Let W be the subset of K^m consisting of the vectors y such that $Q(y) = 0$ for every Q in \mathfrak{A} . Then the projection of $V \subset \mathbf{P}^n(K) \times K^m$ onto the second factor K^m is equal to W .*

To reformulate theorem C, let us consider the ring

$$B = k[X_0, X_1, \dots, X_n; Y_1, \dots, Y_m]$$

together with its subring $B_0 = k[Y_1, \dots, Y_m]$. Denote by B_d the B_0 -module generated in B by the monomials of degree d in X_0, X_1, \dots, X_n . Then $B = \bigoplus_{d \geq 0} B_d$ is a graded ring with J a graded ideal. Define the *graded ring* $A = B/J$ with $A_d = B_d/(B_d \cap J)$. We have the following properties: