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$$u(x)\,u(y) = f(x, y)\,u(xy) \qquad (2)$$

where the factor $f(x, y) \in A$ must satisfy an identity representing the associative law for the triple product $u(x)\,u(y)\,u(z)$; this f was called a "factor set". The extension E is then completely determined by G, the G-module A, and this factor set f. It turns out from the associative law that f is exactly a two dimensional cocycle for G, and that the set of all extensions E of the given G module A by G is exactly the two dimensional cohomology group $H^2(G, A)$. Hence this group, as well as the one dimension cohomology group $H^1(G, A)$, was well known in the 1930's. This made it possible for Eilenberg-Mac Lane and Eckmann to recognize in their papers cited above that the general cohomology of a group includes for dimension 2 the known case of group extensions.

In this description of group extensions by factor sets, the binary operation (of multiplication or perhaps addition) which makes $H^2(G, A)$ a group is given by the multiplication of two factor sets f, f' to form a new factor set

$$f(x, y) f'(x, y) .$$

In his studies of group extension [1934], Baer had raised and answered the question of finding an invariant way of describing this multiplication of two extensions (1) and (1′)—a description independent of the choice of representatives and now called the "Baer product" of extensions. He likewise had considered extensions of a *non-abelian* group A by a group G, and had observed that such an extension, realizing given operators of G on A, are not always possible. Indeed, there is a certain obstruction to forming such an extension, and this obstruction is a three-dimensional cohomology class of $H^3(G, Z)$ where Z is the center of A. This obstruction was identified in this way by Eilenberg-Mac Lane in 1947, and was a central element in the development of the cohomology of groups as an independent subject, not necessarily tied to the motivating topological examples.

5. The Background in Class Field Theory

In the early 20th century, linear algebra was an Anglo-American subject. Hamilton's discovery of quaternions and C. S. Peirce's utilization of idempotents had started the subject off. In 1905 Wedderburn had proved that any finite division algebra was commutative; one year later he proved his structure theorems. In a sense, they reduced the search for all finite dimensional linear algebras to that for all division algebras. Dickson found many, by the study of cyclic algebras. These were algebras of order n^2 over a field K constructed from a cyclic extension N of K of degree n. The crucial ideas of this line of thought were recorded in Dickson's [1923] book "*Algebras and Their Arithmetics*". Its second edition was translated into German in 1927. This translation immediately attracted attention in Germany.

In the early 20th century, algebraic number theory was a Germanic subject. Hilbert's 1898 *Zahlbericht* (where he introduced the term "number ring") had led him to his study of relative quadratic fields. He conjectured that results he found could extend to a general class field theory. This was done 1920-1933 by Takagi, Feutwangler, Artin, Hasse, Chevalley and others. Indeed the development was one of the major driving forces behind the development of abstract algebra in Germany. Part of it dealt with local class field theory (that is, over a field k of p-adic numbers). There one wished in particular to determine over a local field k all the central simple algebras. These (as Brauer, Hasse, and Noether observed in 1932) could all be described as crossed product algebras, as follows. Take a finite normal extension field N of k, with Galois group G. In the vector space E over N of dimension the degree $n = [N\colon k]$ and basis the n elements u_x for $x \in G$ introduce a product by the rule

$$u_x u_y = f(x, y)\, u_{xy}$$

where f is a factor set of G in the multiplicative group N^* of N. With this product, E becomes a central simple algebra over k, and all central simple algebras over a local field, k, can be so represented. On the one hand, this generalizes Dickson's cyclic algebras from his case when the group G is a cyclic one. On the other hand, it describes the possible central simple algebras by factor sets, which in turn are just two-dimensional cocycles of G in N. So here again it is that cohomology enters algebra. In my own case, this is where I first learned of the two-dimensional cohomology group $H^2(G, N^*)$. A long study with Schilling attempting to extend class field theory to non-abelian extensions involved difficulties—and I recall thinking at the time that these difficulties came up because there were no *three-dimensional* factor sets available. Without a topological motivation, Schilling and Mac Lane did not discover the three-dimensional cohomology group $H^3(G, N^*)$. Teichmüller [1940] in a closely related problem about central simple algebras, *did* describe three-dimensional cohomology groups. He did nothing with them (probably because he found the study of complex moduli more fruitful, or perhaps because he was distracted by the war).

Others (Eilenberg-Mac Lane and Eckmann) dutifully cited Teichmüller's results—but it seems unlikely that those results really affected the development.

6. Betti Numbers or Homology Groups

The period 1927-1937 saw an extensive algebraization of combinatorial topology; this process was an essential prerequisite to the cohomology of groups. Before 1927, topology really was combinatorial: a chain in a complex was a string of simplices, each perhaps affected with a multiplicity (a coefficient), and the algebraic manipulation of chains was something auxiliary to their geometric meaning. This is undoubtedly as it must be, at the start; only later can it develop that geometric results follow from long algebraic computations which are not geometrically visible, step by step.

Combinatorial topology, following Poincaré, measured the connectivity of a polyhedron by its Betti numbers and torsion coefficients in each dimension, calculated as they were from chains and their boundaries. Between 1927 and 1934, the style changed completely; now the connectivity was measured by the homology groups, one in each dimension; the invariants of these abelian groups gave the previous Betti numbers and torsion coefficients. It is fascinating to trace this change, as best we now can. I can find no mention of homology *groups* before 1927; for example, the famous 1915 and 1926 papers of Alexander proved the invariance of the Betti *numbers* of a complex, not the invariance of the homology groups. Veblen's *Analysis Situs* (first edition, 1921; second edition, 1931) is all phrased in terms of incidence matrices and Betti numbers, except for one brief section in the back of the book where it is noted that the homology classes module p form a group.

Then in 1927 Vietoris studied the homology of spaces which were not necessarily polyhedra, so that the homotopy groups were not necessarily finitely generated—so of course he (had to) use homology groups. W. Mayer, with references to courses by Vietoris, used homology groups in a 1929 paper on "Abstract Topology" (submitted, November 1927). Heinz Hopf reviewed the paper in the *Jahrbuch*. In his review he notes, evidently with some surprise, that the paper used "group-theoretic methods". E. R. van Kampen's Dutch thesis "Die Combinatorische Topologie und die Dualitits-satz", Den Haag 1929, formulates these ideas by homology *groups*. An influential article by Van der Waerden in 1930 summarized the state of topology: he used homology groups. Alexandroff (whose 1928 papers