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because θ is continuous (I and J are countable), so that A/A' is complete. By considering maximal independent subsets of A and B and observing that only finitely many elements of A are involved in lifting a (finite) basis of F , we see that A/A' has finite rank (similarly for B/B'). As the only finite rank complete groups are f.g. free, it follows that A' and B' are cofinite.

§3. FORMAL GROUPS

DEFINITION. Let \mathcal{A} denote the category of all commutative rings with 1 whose underlying additive group is of the form Z^I , where $\text{card } I \leq \aleph_0$.

Note that $Z[[x_1, \dots, x_n]]$, formal power series over Z in n variables, is an object of \mathcal{A} . Further, \mathcal{A} has an initial object, namely, Z .

LEMMA 9. *Every $A \in \text{obj } \mathcal{A}$ is a complete topological ring in the cofinite topology.*

Proof: By Lemma 1 and Corollary 2, we know A is a complete topological group. It remains to show that multiplication $m: A \times A \rightarrow A$ is continuous, and, for this it suffices to prove the corresponding homomorphism $m': A \otimes A \rightarrow A$ is continuous; this is so because every homomorphism is continuous in the cofinite topology.

The next lemma is taken almost verbatim from [1; p. 12].

LEMMA 10. *If $A \in \text{obj } \mathcal{A}$, then A has a fundamental system of neighborhoods of 0 consisting of cofinite ideals.*

Proof: Let A' be a cofinite subgroup of A . Since multiplication is continuous, there is a cofinite subgroup W of A with $W^2 \subset A'$. Since W is cofinite, it has a f.g. free complement $\langle a_1, \dots, a_r \rangle$. For each j , the continuity of $x \mapsto a_j \cdot x$ at 0 implies the existence of a cofinite $W_j \subset W$ with $a_j W_j \subset A'$. If $U = \bigcap_{j=1}^r W_j$, then U is cofinite in A . Moreover, $a_j U \subset A'$ for all j and $WU \subset A'$ (in fact, $W^2 \subset A'$ and $U \subset W$); hence $AU \subset A'$. Since $1 \in A$, we have $U \subset AU$, so that A/AU is f.g. Now if $(AU)_*$ is the pure subgroup of A generated by AU , then $(AU)_*$ is also an ideal, is cofinite, and $(AU)_* \subset A'_* = A'$ (for A' is already pure).

LEMMA 11. *\mathcal{A} has coproducts.*

Proof: If $A, B \in \text{obj } \mathcal{A}$, define $A \amalg B = (A \otimes B)^\wedge$. Observe that $A \amalg B$ has the correct additive structure, by Corollary 6. By Lemmas 7 and 8,

$$(A \otimes B)^\wedge \cong \varprojlim A \otimes B / (A' \otimes B + A \otimes B') = \varprojlim (A/A' \otimes B/B'),$$

where A' and B' are cofinite subgroups. By Lemma 10, we may assume A' and B' are cofinite ideals. It follows that $A \amalg B$ is a commutative ring with 1, i.e., $A \amalg B \in \text{obj } \mathcal{A}$.

To see that we have a coproduct, consider the diagram

$$\begin{array}{ccccc} & & A \amalg B & & \\ & \nearrow \alpha & \downarrow \gamma' & \nwarrow \beta & \\ A & & & & B \\ & \searrow \alpha' & & \swarrow \beta' & \\ & & C & & \end{array}$$

where $\alpha: a \mapsto a \otimes 1$, $\beta: b \mapsto 1 \otimes b$, $C \in \text{obj } \mathcal{A}$, and α', β' are ring maps. Since $\text{im } \alpha$ and $\text{im } \beta$ lie in $A \otimes B \subset A \amalg B$, the fact that $A \otimes B$ is a coproduct in the category of commutative rings with 1 provides a unique ring map $\gamma: A \otimes B \rightarrow C$ with $\gamma\alpha = \alpha'$ and $\gamma\beta = \beta'$. As C is complete, however, γ has a unique extension $\gamma': A \amalg B \rightarrow C$ making the diagram above commute.

DEFINITION. Let \mathcal{B} be the category of cocommutative Z -coalgebras whose underlying additive group is of the form $Z^{(I)}$, where $\text{card } I \leq \aleph_0$. (N.B. All coalgebras are, by definition, coassociative and have a counit.)

If L is a f.g. Lie ring (i.e., a Lie ring whose additive group is f.g. free), then its universal enveloping algebra is an object of \mathcal{B} . Note also that \mathcal{B} has a final object, namely, Z .

PROPOSITION 12. *There is an antiequivalence of categories $\mathcal{A}^{op} \xrightarrow{\sim} \mathcal{B}$ given by $A \mapsto A^* = \text{Hom}_Z(A, Z)$ taking products to coproducts and final objects to initial objects.*

Proof: By Lemma 3, we know that $A^{**} = A$ (and, if $B \in \text{obj } \mathcal{B}$, then $B^{**} = B$). It remains to consider multiplication $m: A \otimes A \rightarrow A$. As A is complete, we may regard $m: A \amalg A \rightarrow A$. Write $A = B^*$ qua groups. Then Lemma 5 gives

$$A \amalg A = B^* \amalg B^* = (B^* \otimes B^*)^\wedge = (B \otimes B)^*,$$

whence multiplication may be viewed as a map $m: (B \otimes B)^* \rightarrow B^*$. Thus $m^*: B \rightarrow B \otimes B$. This comultiplication is coassociative and cocommutative (because m is associative and commutative). Finally, the unit $u: Z \rightarrow A = B^*$ yields a counit $u^*: B \rightarrow Z$. Thus $B = A^* \in \text{obj } \mathcal{B}$.

The rest of the argument follows as in [1; Chapter I, §13]; we merely give notation and results.

DEFINITION. Let $G\mathcal{B}$ denote the category of all group-objects in \mathcal{B} (call such objects *formal groups over Z*); let $C\mathcal{A}$ denote the category of all cogroup-objects in \mathcal{A} .

LEMMA 13. $A \in \text{obj } C\mathcal{A}$ if and only if A is a commutative Hopf algebra with $A \in \text{obj } \mathcal{A}$; $B \in \text{obj } G\mathcal{B}$ if and only if B is a cocommutative Hopf algebra with $B \in \text{obj } \mathcal{B}$.

N.B. (By Hopf algebra, we mean a Z -bialgebra with antipode.)

We may now state our version of Cartier duality.

THEOREM 14. There is an equivalence of categories $(C\mathcal{A})^{op} \xrightarrow{\sim} G\mathcal{B}$ implemented by $A \mapsto A^* = \text{Hom}_Z(A, Z)$.

Proof: Precisely as in [1], using Proposition 12.

Let us now compare our result with that of Morris and Pareigis [5]. For a commutative ring k , they consider a category $k\text{-Alg}_{pf}$ defined as a certain full subcategory of all commutative topological k -algebras. When $k = Z$, this is their analogue of our category \mathcal{A} . In essence, a commutative topological ring A ($= Z$ -algebra) lies in $Z\text{-Alg}_{pf}$ if $A \cong \varprojlim D_i$, where $\{D_i, p_i^j\}$ is an inverse system with directed index set of discrete commutative rings D_i that are f.g. free as abelian groups and the p_i^j are ring surjections. There is further hypothesis on the inverse system, but suffice it to say that our Z -algebras in \mathcal{A} do lie in $Z\text{-Alg}_{pf}$; moreover, continuity of every ring map in \mathcal{A} shows that \mathcal{A} is a full subcategory of $Z\text{-Alg}_{pf}$. Since $Z\text{-Alg}_{pf}$ may contain algebras of cardinal larger than continuum, \mathcal{A} is genuinely smaller than $Z\text{-Alg}_{pf}$.

In [2], Ditters gives a Cartier duality in which the analogue of \mathcal{A} is called Al_Z : its objects are all commutative topological Z -algebras that are isomorphic to Z^I as a Z -module for some index set I (not necessarily countable) and such that the topology on Z^I is the product topology (each Z being discrete).

THEOREM 15. *The category \mathcal{A} is a proper, full subcategory of the category $\mathbf{Z}\text{-Alg}_{pf}$ of Morris-Pareigis; the category \mathcal{A} is a proper, full subcategory of the category $\mathbf{Al}_{\mathbf{Z}}$ of Ditters.*

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