

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 24 (1978)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** CARTIER DUALITY AND FORMAL GROUPS OVER Z  
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**Kapitel:** §3. Formal Groups  
**DOI:** <https://doi.org/10.5169/seals-49705>

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because  $\theta$  is continuous ( $I$  and  $J$  are countable), so that  $A/A'$  is complete. By considering maximal independent subsets of  $A$  and  $B$  and observing that only finitely many elements of  $A$  are involved in lifting a (finite) basis of  $F$ , we see that  $A/A'$  has finite rank (similarly for  $B/B'$ ). As the only finite rank complete groups are f.g. free, it follows that  $A'$  and  $B'$  are cofinite.

### §3. FORMAL GROUPS

**DEFINITION.** Let  $\mathcal{A}$  denote the category of all commutative rings with 1 whose underlying additive group is of the form  $\mathbb{Z}^I$ , where  $\text{card } I \leq \aleph_0$ .

Note that  $\mathbb{Z}[[x_1, \dots, x_n]]$ , formal power series over  $\mathbb{Z}$  in  $n$  variables, is an object of  $\mathcal{A}$ . Further,  $\mathcal{A}$  has an initial object, namely,  $\mathbb{Z}$ .

**LEMMA 9.** *Every  $A \in \text{obj } \mathcal{A}$  is a complete topological ring in the cofinite topology.*

*Proof:* By Lemma 1 and Corollary 2, we know  $A$  is a complete topological group. It remains to show that multiplication  $m: A \times A \rightarrow A$  is continuous, and, for this it suffices to prove the corresponding homomorphism  $m': A \otimes A \rightarrow A$  is continuous; this is so because every homomorphism is continuous in the cofinite topology.

The next lemma is taken almost verbatim from [1; p. 12].

**LEMMA 10.** *If  $A \in \text{obj } \mathcal{A}$ , then  $A$  has a fundamental system of neighborhoods of 0 consisting of cofinite ideals.*

*Proof:* Let  $A'$  be a cofinite subgroup of  $A$ . Since multiplication is continuous, there is a cofinite subgroup  $W$  of  $A$  with  $W^2 \subset A'$ . Since  $W$  is cofinite, it has a f.g. free complement  $\langle a_1, \dots, a_r \rangle$ . For each  $j$ , the continuity of  $x \mapsto a_j \cdot x$  at 0 implies the existence of a cofinite  $W_j \subset W$  with  $a_j W_j \subset A'$ . If  $U = \bigcap_{j=1}^r W_j$ , then  $U$  is cofinite in  $A$ . Moreover,  $a_j U \subset A'$  for all  $j$  and  $WU \subset A'$  (in fact,  $W^2 \subset A'$  and  $U \subset W$ ); hence  $AU \subset A'$ . Since  $1 \in A$ , we have  $U \subset AU$ , so that  $A/AU$  is f.g. Now if  $(AU)_*$  is the pure subgroup of  $A$  generated by  $AU$ , then  $(AU)_*$  is also an ideal, is cofinite, and  $(AU)_* \subset A'_* = A'$  (for  $A'$  is already pure).

**LEMMA 11.**  *$\mathcal{A}$  has coproducts.*

*Proof:* If  $A, B \in \text{obj } \mathcal{A}$ , define  $A \amalg B = (A \otimes B)^\wedge$ . Observe that  $A \amalg B$  has the correct additive structure, by Corollary 6. By Lemmas 7 and 8,

$$(A \otimes B)^\wedge \cong \varprojlim A \otimes B / (A' \otimes B + A \otimes B') = \varprojlim (A/A' \otimes B/B') ,$$

where  $A'$  and  $B'$  are cofinite subgroups. By Lemma 10, we may assume  $A'$  and  $B'$  are cofinite ideals. It follows that  $A \amalg B$  is a commutative ring with 1, i.e.,  $A \amalg B \in \text{obj } \mathcal{A}$ .

To see that we have a coproduct, consider the diagram

$$\begin{array}{ccccc} & & A \amalg B & & \\ & \alpha \nearrow & \downarrow & \swarrow \beta & \\ A & & \gamma' & & B \\ & \alpha' \searrow & \downarrow & \swarrow \beta' & \\ & & C & & \end{array}$$

where  $\alpha: a \mapsto a \otimes 1$ ,  $\beta: b \mapsto 1 \otimes b$ ,  $C \in \text{obj } \mathcal{A}$ , and  $\alpha', \beta'$  are ring maps. Since  $\text{im } \alpha$  and  $\text{im } \beta$  lie in  $A \otimes B \subset A \amalg B$ , the fact that  $A \otimes B$  is a coproduct in the category of commutative rings with 1 provides a unique ring map  $\gamma: A \otimes B \rightarrow C$  with  $\gamma\alpha = \alpha'$  and  $\gamma\beta = \beta'$ . As  $C$  is complete, however,  $\gamma$  has a unique extension  $\gamma': A \amalg B \rightarrow C$  making the diagram above commute.

**DEFINITION.** Let  $\mathcal{B}$  be the category of cocommutative  $Z$ -coalgebras whose underlying additive group is of the form  $Z^{(I)}$ , where  $\text{card } I \leq \aleph_0$ . (N.B. All coalgebras are, by definition, coassociative and have a counit.)

If  $L$  is a f.g. Lie ring (i.e., a Lie ring whose additive group is f.g. free), then its universal enveloping algebra is an object of  $\mathcal{B}$ . Note also that  $\mathcal{B}$  has a final object, namely,  $Z$ .

**PROPOSITION 12.** *There is an antiequivalence of categories  $\mathcal{A}^{op} \xrightarrow{\sim} \mathcal{B}$  given by  $A \mapsto A^* = \text{Hom}_Z(A, Z)$  taking products to coproducts and final objects to initial objects.*

*Proof:* By Lemma 3, we know that  $A^{**} = A$  (and, if  $B \in \text{obj } \mathcal{B}$ , then  $B^{**} = B$ ). It remains to consider multiplication  $m: A \otimes A \rightarrow A$ . As  $A$  is complete, we may regard  $m: A \amalg A \rightarrow A$ . Write  $A = B^*$  qua groups. Then Lemma 5 gives

$$A \amalg A = B^* \amalg B^* = (B^* \otimes B^*)^\wedge = (B \otimes B)^* ,$$

whence multiplication may be viewed as a map  $m: (B \otimes B)^* \rightarrow B^*$ . Thus  $m^*: B \rightarrow B \otimes B$ . This comultiplication is coassociative and cocommutative (because  $m$  is associative and commutative). Finally, the unit  $u: Z \rightarrow A = B^*$  yields a counit  $u^*: B \rightarrow Z$ . Thus  $B = A^* \in \text{obj } \mathcal{B}$ .

The rest of the argument follows as in [1; Chapter I, §13]; we merely give notation and results.

**DEFINITION.** Let  $G\mathcal{B}$  denote the category of all group-objects in  $\mathcal{B}$  (call such objects *formal groups over  $Z$* ); let  $C\mathcal{A}$  denote the category of all cogroup-objects in  $\mathcal{A}$ .

**LEMMA 13.**  $A \in \text{obj } C\mathcal{A}$  if and only if  $A$  is a commutative Hopf algebra with  $A \in \text{obj } \mathcal{A}$ ;  $B \in \text{obj } G\mathcal{B}$  if and only if  $B$  is a cocommutative Hopf algebra with  $B \in \text{obj } \mathcal{B}$ .

N.B. (By Hopf algebra, we mean a  $Z$ -bialgebra with antipode.)

We may now state our version of Cartier duality.

**THEOREM 14.** There is an equivalence of categories  $(C\mathcal{A})^{op} \xrightarrow{\sim} G\mathcal{B}$  implemented by  $A \mapsto A^* = \text{Hom}_Z(A, Z)$ .

*Proof:* Precisely as in [1], using Proposition 12.

Let us now compare our result with that of Morris and Pareigis [5]. For a commutative ring  $k$ , they consider a category  $k\text{-Alg}_{pf}$  defined as a certain full subcategory of all commutative topological  $k$ -algebras. When  $k = Z$ , this is their analogue of our category  $\mathcal{A}$ . In essence, a commutative topological ring  $A$  ( $= Z$ -algebra) lies in  $Z\text{-Alg}_{pf}$  if  $A \cong \varprojlim D_i$ , where  $\{D_i, p_i^j\}$  is an inverse system with directed index set of discrete commutative rings  $D_i$  that are f.g. free as abelian groups and the  $p_i^j$  are ring surjections. There is further hypothesis on the inverse system, but suffice it to say that our  $Z$ -algebras in  $\mathcal{A}$  do lie in  $Z\text{-Alg}_{pf}$ ; moreover, continuity of every ring map in  $\mathcal{A}$  shows that  $\mathcal{A}$  is a full subcategory of  $Z\text{-Alg}_{pf}$ . Since  $Z\text{-Alg}_{pf}$  may contain algebras of cardinal larger than continuum,  $\mathcal{A}$  is genuinely smaller than  $Z\text{-Alg}_{pf}$ .

In [2], Ditters gives a Cartier duality in which the analogue of  $\mathcal{A}$  is called  $Al_Z$ : its objects are all commutative topological  $Z$ -algebras that are isomorphic to  $Z^I$  as a  $Z$ -module for some index set  $I$  (not necessarily countable) and such that the topology on  $Z^I$  is the product topology (each  $Z$  being discrete).

THEOREM 15. *The category  $\mathcal{A}$  is a proper, full subcategory of the category  $Z\text{-Alg}_{pf}$  of Morris-Pareigis; the category  $\mathcal{A}$  is a proper, full subcategory of the category  $Al_Z$  of Ditters.*

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(Reçu le 27 juillet 1977)

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