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because  $\theta$  is continuous (I and J are countable), so that A/A' is complete. By considering maximal independent subsets of A and B and observing that only finitely many elements of A are involved in lifting a (finite) basis of F, we see that A/A' has finite rank (similarly for B/B'). As the only finite rank complete groups are f.g. free, it follows that A' and B' are cofinite.

# §3. FORMAL GROUPS

DEFINITION. Let  $\mathscr{A}$  denote the category of all commutative rings with 1 whose underlying additive group is of the form  $Z^I$ , where card  $I \leq \aleph_0$ .

Note that  $Z[[x_1, ..., x_n]]$ , formal power series over Z in n variables, is an object of  $\mathscr{A}$ . Further,  $\mathscr{A}$  has an initial object, namely, Z.

LEMMA 9. Every  $A \in obj \mathscr{A}$  is a complete topological ring in the cofinite topology.

*Proof*: By Lemma 1 and Corollary 2, we know A is a complete topological group. It remains to show that multiplication  $m: A \times A \to A$  is continuous, and, for this it suffices to prove the corresponding homomorphism  $m': A \otimes A \to A$  is continuous; this is so because every homomorphism is continuous in the cofinite topology.

The next lemma is taken almost verbatim from [1; p. 12].

LEMMA 10. If  $A \in obj \mathcal{A}$ , then A has a fundamental system of neighborhoods of 0 consisting of cofinite ideals.

*Proof*: Let A' be a cofinite subgroup of A. Since multiplication is continuous, there is a cofinite subgroup W of A with  $W^2 \subset A'$ . Since W is cofinite, it has a f.g. free complement  $\langle a_1, ..., a_r \rangle$ . For each j, the continuity of  $x \mapsto a_j \cdot x$  at 0 implies the existence of a cofinite  $W_j \subset W$  with  $a_j W_j \subset A'$ . If  $U = \bigcap_{j=1}^r W_j$ , then U is cofinite in A. Moreover,  $a_j U \subset A'$  for all j and  $WU \subset A'$  (in fact,  $W^2 \subset A'$  and  $U \subset W$ ); hence  $AU \subset A'$ . Since  $1 \in A$ , we have  $U \subset AU$ , so that A/AU is f.g. Now if  $(AU)_*$  is the pure subgroup of A generated by AU, then  $(AU)_*$  is also an ideal, is cofinite, and  $(AU)_* \subset A'_* = A'$  (for A' is already pure).

LEMMA 11. A has coproducts.

*Proof*: If  $A, B \in \text{obj } \mathcal{A}$ , define  $A \coprod B = (A \otimes B)^{\wedge}$ . Observe that  $A \coprod B$  has the correct additive structure, by Corollary 6. By Lemmas 7 and 8,

$$(A \otimes B)^{\wedge} \cong \lim_{A \to B} A \otimes B/(A' \otimes B + A \otimes B') = \lim_{A \to B} (A/A' \otimes B/B'),$$

where A' and B' are cofinite subgroups. By Lemma 10, we may assume A' and B' are cofinite ideals. It follows that  $A \coprod B$  is a commutative ring with 1, i.e.,  $A \coprod B \in \text{obj } \mathcal{A}$ .

To see that we have a coproduct, consider the diagram



where  $\alpha: a \mapsto a \otimes 1$ ,  $\beta: b \mapsto 1 \otimes b$ ,  $C \in obj \mathscr{A}$ , and  $\alpha', \beta'$  are ring maps. Since im  $\alpha$  and im  $\beta$  lie in  $A \otimes B \subset A \coprod B$ , the fact that  $A \otimes B$  is a coproduct in the category of commutative rings with 1 provides a unique ring map  $\gamma: A \otimes B \to C$  with  $\gamma \alpha = \alpha'$  and  $\gamma \beta = \beta'$ . As C is complete, however,  $\gamma$  has a unique extension  $\gamma': A \coprod B \to C$  making the diagram above commute.

DEFINITION. Let  $\mathscr{B}$  be the category of cocommutative Z-coalgebras whose underlying additive group is of the form  $Z^{(I)}$ , where card  $I \leq \aleph_0$ . (N.B. All coalgebras are, by definition, coassociative and have a counit.)

If L is a f.g. Lie ring (i.e., a Lie ring whose additive group is f.g. free), then its universal enveloping algebra is an object of  $\mathcal{B}$ . Note also that  $\mathcal{B}$  has a final object, namely, Z.

PROPOSITION 12. There is an antiequivalence of categories  $\mathscr{A}^{op} \xrightarrow{\sim} \mathscr{B}$  given by  $A \mapsto A^* = \operatorname{Hom}_Z(A, Z)$  taking products to coproducts and final objects to initial objects.

*Proof*: By Lemma 3, we know that  $A^{**} = A$  (and, if  $B \in \text{obj } \mathcal{B}$ , then  $B^{**} = B$ ). It remains to consider multiplication  $m: A \otimes A \to A$ . As A is complete, we may regard  $m: A \coprod A \to A$ . Write  $A = B^*$  qua groups. Then Lemma 5 gives

 $A \coprod A = B^* \coprod B^* = (B^* \otimes B^*)^{\wedge} = (B \otimes B)^*,$ 

whence multiplication may be viewed as a map  $m: (B \otimes B)^* \to B^*$ . Thus  $m^*: B \to B \otimes B$ . This comultiplication is coassociative and cocommutative (because *m* is associative and commutative). Finally, the unit  $u: Z \to A = B^*$  yields a counit  $u^*: B \to Z$ . Thus  $B = A^* \in \text{obj } \mathcal{B}$ .

The rest of the argument follows as in [1; Chapter I, §13]; we merely give notation and results.

DEFINITION. Let  $G\mathscr{B}$  denote the category of all group-objects in  $\mathscr{B}$  (call such objects *formal groups over* Z); let  $C\mathscr{A}$  denote the category of all cogroup-objects in  $\mathscr{A}$ .

LEMMA 13.  $A \in \text{obj } C\mathcal{A}$  if and only if A is a commutative Hopf algebra with  $A \in \text{obj } \mathcal{A}$ ;  $B \in \text{obj } G\mathcal{B}$  if and only if B is a cocommutative Hopf algebra with  $B \in \text{obj } \mathcal{B}$ .

N.B. (By Hopf algebra, we mean a Z-bialgebra with antipode.)

We may now state our version of Cartier duality.

THEOREM 14. There is an equivalence of categories  $(C\mathscr{A})^{o_p} \xrightarrow{\sim} G\mathscr{B}$ implemented by  $A \mapsto A^* = \operatorname{Hom}_Z(A, Z)$ .

*Proof*: Precisely as in [1], using Proposition 12.

Let us now compare our result with that of Morris and Pareigis [5]. For a commutative ring k, they consider a category k-Alg<sub>pf</sub> defined as a certain full subcategory of all commutative topological k-algebras. When k = Z, this is their analogue of our category  $\mathscr{A}$ . In essence, a commutative topological ring A (=Z-algebra) lies in Z-Alg<sub>pf</sub> if  $A \cong \lim_{i \to i} D_i$ , where  $\{D_i, p_i^j\}$  is an inverse system with directed index set of discrete commutative rings  $D_i$  that are f.g. free as abelian groups and the  $p_i^j$  are ring surjections. There is further hypothesis on the inverse system, but suffice it to say that our Z-algebras in  $\mathscr{A}$  do lie in Z-Alg<sub>pf</sub>; moreover, continuity of every ring map in  $\mathscr{A}$  shows that  $\mathscr{A}$  is a full subcategory of Z-Alg<sub>pf</sub>. Since Z-Alg<sub>pf</sub> may contain algebras of cardinal larger than continuum,  $\mathscr{A}$  is genuinely smaller than Z-Alg<sub>pf</sub>.

In [2], Ditters gives a Cartier duality in which the analogue of  $\mathscr{A}$  is called  $Al_Z$ : its objects are all commutative topological Z-algebras that are isomorphic to  $Z^I$  as a Z-module for some index set I (not necessarily countable) and such that the topology on  $Z^I$  is the product topology (each Z being discrete).

THEOREM 15. The category  $\mathscr{A}$  is a proper, full subcategory of the category Z-Alg<sub>pf</sub> of Morris-Pareigis; the category  $\mathscr{A}$  is a proper, full subcategory of the category  $Al_Z$  of Ditters.

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