§3. Formal Groups

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because θ is continuous (I and J are countable), so that A/A' is complete. By considering maximal independent subsets of A and B and observing that only finitely many elements of A are involved in lifting a (finite) basis of F, we see that A/A' has finite rank (similarly for B/B'). As the only finite rank complete groups are f.g. free, it follows that A' and B' are cofinite.

§3. Formal Groups

Definition. Let $\mathscr{A}$ denote the category of all commutative rings with 1 whose underlying additive group is of the form Z^I, where card $I \leqslant \aleph_0$.

Note that $Z[[x_1, ..., x_n]]$, formal power series over Z in n variables, is an object of $\mathscr{A}$. Further, $\mathscr{A}$ has an initial object, namely, Z.

Lemma 9. *Every $A \in \operatorname{obj} \mathscr{A}$ is a complete topological ring in the cofinite topology.*

Proof: By Lemma 1 and Corollary 2, we know A is a complete topological group. It remains to show that multiplication $m: A \times A \to A$ is continuous, and, for this it suffices to prove the corresponding homomorphism $m': A \otimes A \to A$ is continuous; this is so because every homomorphism is continuous in the cofinite topology.

The next lemma is taken almost verbatim from [1; p. 12].

Lemma 10. *If $A \in \operatorname{obj} \mathscr{A}$, then A has a fundamental system of neighborhoods of 0 consisting of cofinite ideals.*

Proof: Let A' be a cofinite subgroup of A. Since multiplication is continuous, there is a cofinite subgroup W of A with $W^2 \subset A'$. Since W is cofinite, it has a f.g. free complement $<a_1, ..., a_r>$. For each j, the continuity of $x \mapsto a_j \cdot x$ at 0 implies the existence of a cofinite $W_j \subset W$ with $a_j W_j \subset A'$. If $U = \bigcap_{j=1}^{r} W_j$, then U is cofinite in A. Moreover, $a_j U \subset A'$ for all j and $WU \subset A'$ (in fact, $W^2 \subset A'$ and $U \subset W$); hence $AU \subset A'$. Since $1 \in A$, we have $U \subset AU$, so that A/AU is f.g. Now if $(AU)_*$ is the pure subgroup of A generated by AU, then $(AU)_*$ is also an ideal, is cofinite, and $(AU)_* \subset A'_* = A'$ (for A' is already pure).

Lemma 11. *$\mathscr{A}$ has coproducts.*

Proof: If $A, B \in \operatorname{obj} \mathscr{A}$, define $A \amalg B = (A \otimes B)^{\wedge}$. Observe that $A \amalg B$ has the correct additive structure, by Corollary 6. By Lemmas 7 and 8,

$$(A \otimes B)^{\wedge} \cong \varprojlim A \otimes B/(A' \otimes B + A \otimes B') = \varprojlim (A/A' \otimes B/B'),$$

where A' and B' are cofinite subgroups. By Lemma 10, we may assume A' and B' are cofinite ideals. It follows that $A \amalg B$ is a commutative ring with 1, i.e., $A \amalg B \in \operatorname{obj} \mathscr{A}$.

To see that we have a coproduct, consider the diagram

$$\begin{tikzcd} & A \amalg B \arrow[dd, dashed, "\gamma'"] & \\ A \arrow[ur, "\alpha"] \arrow[dr, "\alpha'"'] & & B \arrow[ul, "\beta"'] \arrow[dl, "\beta'"] \\ & C & \end{tikzcd}$$

where $\alpha: a \mapsto a \otimes 1$, $\beta: b \mapsto 1 \otimes b$, $C \in \operatorname{obj} \mathscr{A}$, and α', β' are ring maps. Since $\operatorname{im} \alpha$ and $\operatorname{im} \beta$ lie in $A \otimes B \subset A \amalg B$, the fact that $A \otimes B$ is a coproduct in the category of commutative rings with 1 provides a unique ring map $\gamma: A \otimes B \to C$ with $\gamma\alpha = \alpha'$ and $\gamma\beta = \beta'$. As C is complete, however, γ has a unique extension $\gamma': A \amalg B \to C$ making the diagram above commute.

DEFINITION. Let $\mathscr{B}$ be the category of cocommutative Z-coalgebras whose underlying additive group is of the form $Z^{(I)}$, where card $I \leqslant \aleph_0$. (N.B. All coalgebras are, by definition, coassociative and have a counit.)

If L is a f.g. Lie ring (i.e., a Lie ring whose additive group is f.g. free), then its universal enveloping algebra is an object of $\mathscr{B}$. Note also that $\mathscr{B}$ has a final object, namely, Z.

PROPOSITION 12. *There is an antiequivalence of categories* $\mathscr{A}^{op} \xrightarrow{\sim} \mathscr{B}$ *given by* $A \mapsto A^* = \operatorname{Hom}_Z(A, Z)$ *taking products to coproducts and final objects to initial objects.*

Proof: By Lemma 3, we know that $A^{**} = A$ (and, if $B \in \operatorname{obj} \mathscr{B}$, then $B^{**} = B$). It remains to consider multiplication $m: A \otimes A \to A$. As A is complete, we may regard $m: A \amalg A \to A$. Write $A = B^*$ qua groups. Then Lemma 5 gives

$$A \amalg A = B^* \amalg B^* = (B^* \otimes B^*)^{\wedge} = (B \otimes B)^*,$$

whence multiplication may be viewed as a map $m: (B \otimes B)^* \to B^*$. Thus $m^*: B \to B \otimes B$. This comultiplication is coassociative and cocommutative (because m is associative and commutative). Finally, the unit $u: Z \to A = B^*$ yields a counit $u^*: B \to Z$. Thus $B = A^* \in \text{obj } \mathscr{B}$.

The rest of the argument follows as in [1; Chapter I, §13]; we merely give notation and results.

Definition. Let $G\mathscr{B}$ denote the category of all group-objects in $\mathscr{B}$ (call such objects *formal groups over* Z); let $C\mathscr{A}$ denote the category of all cogroup-objects in $\mathscr{A}$.

Lemma 13. *$A \in \text{obj } C\mathscr{A}$ if and only if A is a commutative Hopf algebra with $A \in \text{obj } \mathscr{A}$; $B \in \text{obj } G\mathscr{B}$ if and only if B is a cocommutative Hopf algebra with $B \in \text{obj } \mathscr{B}$.*

N.B. (By Hopf algebra, we mean a Z-bialgebra with antipode.)

We may now state our version of Cartier duality.

Theorem 14. *There is an equivalence of categories $(C\mathscr{A})^{op} \xrightarrow{\sim} G\mathscr{B}$ implemented by $A \mapsto A^* = \mathrm{Hom}_Z(A, Z)$.*

Proof: Precisely as in [1], using Proposition 12.

Let us now compare our result with that of Morris and Pareigis [5]. For a commutative ring k, they consider a category $k\text{-Alg}_{pf}$ defined as a certain full subcategory of all commutative topological k-algebras. When $k = Z$, this is their analogue of our category $\mathscr{A}$. In essence, a commutative topological ring A ($=Z$-algebra) lies in $Z\text{-Alg}_{pf}$ if $A \cong \varprojlim D_i$, where $\{D_i, p_i^j\}$ is an inverse system with directed index set of discrete commutative rings D_i that are f.g. free as abelian groups and the p_i^j are ring surjections. There is further hypothesis on the inverse system, but suffice it to say that our Z-algebras in $\mathscr{A}$ do lie in $Z\text{-Alg}_{pf}$; moreover, continuity of every ring map in $\mathscr{A}$ shows that $\mathscr{A}$ is a full subcategory of $Z\text{-Alg}_{pf}$. Since $Z\text{-Alg}_{pf}$ may contain algebras of cardinal larger than continuum, $\mathscr{A}$ is genuinely smaller than $Z\text{-Alg}_{pf}$.

In [2], Ditters gives a Cartier duality in which the analogue of $\mathscr{A}$ is called Al_Z: its objects are all commutative topological Z-algebras that are isomorphic to Z^I as a Z-module for some index set I (not necessarily countable) and such that the topology on Z^I is the product topology (each Z being discrete).

THEOREM 15. *The category* $\mathscr{A}$ *is a proper, full subcategory of the category* Z-Alg_{pf} *of Morris-Pareigis; the category* $\mathscr{A}$ *is a proper, full subcategory of the category* Al_Z *of Ditters.*

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