Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 24 (1978)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INVARIANTS OF FINITE REFLECTION GROUPS

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Kapitel: 2. Mean Value Properties

DOI: https://doi.org/10.5169/seals-49704

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By lemma 3.4 $P^*(x) = \Pi(x) i(x)$, where i is a homogeneous invariant. If deg i > 0, then $P^* \in \mathscr{I} \Rightarrow P \in \mathscr{I}$. Otherwise $P^* = c \Pi$, c a constant. By assumption $P(\mathfrak{d}) \Pi = 0$, while $a(\mathfrak{d}) \Pi = 0$ for $a \in \mathscr{I}$. It follows that $P^*(\mathfrak{d}) \Pi = c(\Pi, \Pi) \Rightarrow c = 0$, so that $P \equiv 0 \pmod{\mathscr{I}}$.

2. MEAN VALUE PROPERTIES

We prove the equivalence of system (4.1) and a certain mean value property.

Theorem 4.3 (Steinberg [21]). Let $f(x) \in C$ in the n-dimensional region \mathcal{R} and let it satisfy the mean value property (m.v.p.)

$$(4.6) f(x) = \frac{1}{|G|} \sum_{\sigma \in G} f(x + \sigma y), x \in \mathcal{R} \text{ and } ||y|| < \varepsilon_x,$$

where $\inf_{x \in K} \varepsilon_x > 0$ for any compact subset K of \mathcal{R} and $||y||^2 = \sum_{i=1}^n y_i^2$. This m.v.p. is equivalent to having $f \in C^{\infty}$ and satisfying (4.1). It follows from Theorem 4.2 that the space S of continuous solutions to (4.6) = $D \Pi$.

REMARK. The harmonic functions on \mathcal{R} are characterized as the continuous functions on \mathcal{R} satisfying the m.v.p. $f(x) = \int f(x+y) d\sigma(y)$, $x \in \mathcal{R}$ and $||y|| < \varepsilon_{x'}$ where $d\sigma(y)$ is the normalized Haar measure on the orthogonal group O(n). (4.6) is just the G-analog of this m.v.p.

Proof of Theorem 4.3. Suppose first that f(x) is C^{∞} on \mathcal{R} and satisfies (4.6). Let a(x) be any homogeneous invariant of positive degree. Apply the operator $a(\partial_{\nu})$ to both sides of (4.6). In view of Lemma 4.1, we get

(4.7)
$$0 = a(\partial_{y}) f(x) = \frac{1}{|G|} \sum_{\sigma \in G} a(\partial_{y}) f(x + \sigma y)$$
$$= \frac{1}{|G|} \sum_{\sigma \in G} [a(\partial_{y}) f(x + y)] (\sigma y)$$

Use $a(\delta_y) f(x+y) = a(\delta_x) f(x+y)$ and set y = 0. We obtain $a(\delta_x) f(x) = 0$, $x \in \mathcal{R}$ and a any homogeneous invariant of positive degree. Hence $a(\delta_x) f(x) = 0$, $x \in \mathcal{R}$ and $a \in \mathcal{I}$. Since $\sum_{i=1}^{n} x_i^2 \in \mathcal{I}$, we conclude in particular that f(x) is harmonic on \mathcal{R} .

Suppose next that f(x) is C on \mathcal{R} and satisfies (4.6). Let $\{\delta_k\}$ be a sequence of C^{∞} functions on R^n such that $\int \delta_k(x) dx = 1$, support of $\delta_k = \left\{ x \mid \|x\| \leqslant \frac{1}{k} \right\}$, $\delta_k(x) \geqslant 0$ for all x and k. Let

$$f_k(x) = \int f(x-y) \, \delta_k(y) \, dy = \int f(y) \, \delta_k(x-y) \, dy.$$

It is readily checked that for any compact subset S of \mathcal{R} , $f_k(x) \in C^{\infty}$ on Int S (= interior of S) and satisfies (4.6) with \mathcal{R} replaced by Int S, provided k is sufficiently large, and $f_k \to f$ uniformly on S as $k \to \infty$. For k sufficiently large, f_k is harmonic on Int S. It follows from Harnack's Theorem ([15], p. 248) that f(x) is harmonic on \mathcal{R} . Hence f(x) is real analytic on \mathcal{R} ([15], p. 251) and so certainly C^{∞} on \mathcal{R} .

Conversely let $f \in C^{\infty}$ on \mathcal{R} and a (d) f = 0, $x \in \mathcal{R}$ and $a \in \mathcal{I}$. Then f is harmonic and so real analytic on \mathcal{R} . Hence there exists $\varepsilon_x > 0$ such that

$$f(x+y) = \sum_{m=0}^{\infty} \frac{1}{m!} (\partial_x, y)^m f(x), x \in \mathcal{R}$$

and $||y|| < \varepsilon_x$. It follows that

(4.8)
$$\frac{1}{|G|} \sum_{\sigma \in G} f(x + \sigma y) = \sum_{m=0}^{\infty} \frac{P_m(\partial_x, y)}{m!} f(x), x \in \mathcal{R}$$

and $||y|| < \varepsilon_x$ where

(4.9)
$$P_m(x,y) = \frac{1}{|G|} \sum_{\sigma \in G} (x,\sigma y)^m = \frac{1}{|G|} \sum_{\sigma \in G} (\sigma x,y)^m.$$

From (4.9), we see that for fixed y, each $P_m(x, y)$ is a homogeneous invariant polynomial in x of degree m. It follows that $P_m(\delta_x, y) f(x) = 0$, $x \in \mathcal{R}$ and $m \le 1$, and (4.8) reduces to (4.6).

The solution space to either (4.1) or (4.6) is the finite dimensional vector space $D \Pi$. The following result gives further information on $D \Pi$.

Theorem 4.4 (Chevalley [4]). Let $S_m = vector space of homogeneous$ polynomials of degree m in D Π , $0 \le m < \infty$, so that D $\Pi = \sum_{m=0}^{\infty} \oplus S_m$. Let $d_1, ..., d_n$ be the degrees of the basic homogeneous invariants for G. Then

(4.10)
$$\sum_{m=0}^{\infty} (\dim S_m) t^m = \prod_{i=1}^{n} \frac{1 - t^{d_i}}{1 - t}$$

and dim $D \Pi = |G|$.

We prove first the preliminary

LEMMA 4.2. Let $R = k [x_1, ..., x_n] = \text{ring of polynomials in } x_1, ..., x_n$ with coefficients from k, k being any field of characteristic 0. Let G be a finite reflection group acting on k^n and \mathcal{I} the ideal generated by homogeneous invariants of positive degree. For any polynomial P, let \overline{P} be its residue class in the residue class ring R/\mathcal{I} . Suppose that $P_1, ..., P_s$ are homogeneous polynomials such that $\overline{P}_1, ..., \overline{P}_s$ are linearly independent over R/\mathcal{I} (the latter is a vector space over k). Then $P_1, ..., P_s$ are linearly independent over k(I), the field obtained by adjoining the set I of all invariant polynomials to k.

Proof. Suppose $\sum_{i=1}^{s} V_i P_i = 0$ where $V_i \in k(I)$, $1 \le i \le s$. We may suppose that the V_i' s are homogeneous and $[\deg V_i + \deg P_i]$ is the same for all i. Let $I_1, ..., I_n$ be a basic set of homogeneous invariants of positive degree. Let S_j , $0 \le j < \infty$, be the different monomials in $I_1 ... I_n$ arranged by increasing x-degree, with $s_0 = 1$. Let $V_i = \sum_{j=0}^{\infty} k_{ij} S_j$, $1 \le i \le s$, the k_{ij}' s being elements of k, and define k_{i0} to be 0. We have

(4.11)
$$\sum_{i=1}^{s} V_{i} P_{i} = \sum_{j=0}^{\infty} \left[\sum_{i=1}^{s} k_{ij} P_{i} \right] S_{j} = 0$$

Assume, as induction hypothesis, that $k_{ij} = 0$ for j < l. Thus $\sum_{j=l}^{\infty} \left[\sum_{i=1}^{s} k_{ij} P_i\right] S_j = 0$. $S_i \notin$ ideal generated by the S_j' s, j > l, as $I_1, ..., I_n$ are algebraically independent. It follows from Lemma 2.1 that $\sum_{i=1}^{s} k_{il} P_i \in \mathscr{I} \Leftrightarrow \sum_{i=1}^{s} k_{il} \overline{P}_i = 0 \Leftrightarrow k_{il} = 0, 1 \leqslant i \leqslant s$. Hence all $k_{ij} = 0$ and $V_i = 0, 1 \leqslant i \leqslant s$. I.e. $P_1, ..., P_s$ are linearly independent over k(I).

We now return to the proof of Theorem 4.4. Let $A_1, ..., A_q$ be homogeneous polynomials such that $\overline{A}_1, ..., \overline{A}_q$ form a basis for R/\mathscr{I} . By induction on the degree, we see that every polynomial P may be expressed as

$$(4.12) P = \sum_{i=1}^{q} J_i A_i$$

where the J_i 's are invariant polynomials. Lemma 4.2 shows that this representation is unique. Let $R_m = \text{set}$ of homogeneous polynomials of degree m, $I_m = I \cap R_m$, $(R/\mathcal{I})_m = \text{vector}$ space spanned by those \overline{A}_i 's for which degree $A_i = m$. Let

$$\mathfrak{p}_{R}(t) = \sum_{m=0}^{\infty} (\dim R_{m}) t^{m}, \quad \mathfrak{p}_{I}(t) = \sum_{m=0}^{\infty} (\dim I_{m}) t^{m},$$

$$\mathfrak{p}_{R\mathscr{I}}(t) = \sum_{m=0}^{\infty} \dim (R/\mathscr{I})_{m} t^{m}.$$

In view of the uniqueness of the representation (4.12), we have

$$\mathfrak{p}_{R}(t) = \mathfrak{p}_{I}(t) \mathfrak{p}_{R/\mathscr{I}}(t)$$

Now

$$\mathfrak{p}_{I}(t) = \frac{1}{\prod_{i=1}^{n} (1 - t^{d_{i}})} \quad \text{(formula (2.5))}$$

while

$$\mathfrak{p}_R(t) = \frac{1}{(1-t)^n}$$

(as dim $R_m = \binom{m+n-1}{m}$). By Fischer's Theorem R/\mathscr{I} may be identified with $D\Pi$, so that $\mathfrak{p}_{R/\mathscr{I}}(t) = \sum_{m=0}^{\infty} (\dim S_m) t^m$. Thus (4.13) becomes (4.10).

Set t = 1 in (4.10). The left side becomes $\sum_{m=0}^{\infty} \dim S_m = \dim D \Pi$. Since

$$\frac{1-t^{a_i}}{1-t}=1+t+\ldots+t^{d_{i-1}}=d_i$$

at t = 1, the right side becomes $\prod_{i=1}^{n} d_i = |G|$ (by Theorem 2.2). Thus dim $D \Pi = |G|$.

We now describe the solution space to (4.6) when we restrict the direction of y. For simplicity, we restrict ourselves to irreducible groups (the reducible case is discussed in [12]).

THEOREM 4.5. Let $f(x) \in C$ in the n-dimensional region \mathcal{R} and satisfy the m.v.p.

$$(4.14) f(x) = \frac{1}{|G|} \sum_{\sigma \in G} f(x + t \sigma y), x \in \mathcal{R} \text{ and } 0 < t < \varepsilon_x,$$

inf $\varepsilon_x > 0$ for any compact subset K of \mathcal{R} and y denoting a fixed vector $\varphi = 0$. This m.v.p. is equivalent to having $f \in C^{\infty}$ on \mathcal{R} and $P_m(\mathfrak{d}_x, y)$ $f = 0, x \in \mathcal{R}$ and $1 \leqslant m < \infty$, P_m being defined by (4.9).

Proof. Suppose first that $f \in C^{\infty}$ on \mathcal{R} and satisfies (4.14). Using the finite Taylor expansion for $f(x+t\sigma y)$, we get for each integer $N \ge 0$

(4.15)
$$0 = \sum_{m=1}^{N} \left[\frac{P_m(\hat{\partial}_x, y) f}{m!} \right] t^m + O(t^{N+1}) \text{ as } t \to 0.$$

Dividing by successive powers of t and letting $t \to 0$, we conclude $P_m(\delta_x, y) f = 0$, $x \in \mathcal{R}$ and $1 \le m < \infty$. If $f \in C$, then we argue as in the proof of Theorem 4.3, introducing the functions f_k . For any compact subset S of \mathcal{R} and k sufficiently large, the f_k 's will be C^{∞} on Int S and satisfy there $P_m(\delta_x, y) f = 0$, $1 \le m < \infty$. $P_2(x, y)$ is a non-zero homogeneous invariant of degree 2. For irreducible G, there is up to a multiplicative constant, only one such invariant, namely $\sum_{i=1}^{n} x_i^2$. Thus

 $P_2(x, y) = c(y) \sum_{i=1}^{n} x_i^2$, where $c(y) \neq 0$ is a constant depending on y. Thus for k sufficiently large, $f_k(x)$ is harmonic on Int S. Since $f_k \to f$ uniformly on compact subsets of \mathcal{R} , f(x) is harmonic on \mathcal{R} and hence certainly

Conversely, let $P_m(\delta_x, y) f = 0$, $x \in \mathcal{R}$ and $1 \le m < \infty$. Since $P_2(\delta_x, y) f = 0$, f is harmonic and so real analytic on \mathcal{R} . It follows that there exists $\varepsilon_x > 0$ such that

$$(4.16) \qquad \frac{1}{\mid G \mid} \sum_{\sigma \in G} f(x + t \sigma y) = \sum_{m=0}^{\infty} \left[\frac{P_m(\partial_x, y) f}{m!} \right] t^m, \ x \in \mathcal{R}$$

and $0 < t < \varepsilon_x$.

 C^{∞} on \mathcal{R} .

Since $P_m(\delta_x, y) f = 0$, $x \in \mathcal{R}$ and $1 \le m < \infty$, (4.16) reduces to (4.14). We shall describe the solution space to $P_m(\delta_x, y) f = 0, 1 \le m < \infty$, y being a fixed vector $\neq 0$. We first prove some preliminary lemmas.

LEMMA 4.3. Let \mathscr{C} be a collection of homogeneous polynomials in $k [x_1 ..., x_n]$ of positive degree, k being a field of characteristic 0. Let G be a finite reflection group acting on k^n . The following conditions are equivalent.

i) \mathscr{C} is a basis for the invariants of G

- ii) $\mathscr C$ is a basis for the ideal $\mathscr I$ generated by the homogeneous invariants of positive degree.
- iii) Let $d_1, ..., d_n$ be the degrees of the basic homogeneous invariants of G.

For each d_i there exists a polynomial $P_i \in \mathcal{C}$ of degree d_i such that

$$\frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} \neq 0.$$

Proof. Let $\mathscr{I}(\mathscr{C})=$ ideal generated by \mathscr{C} , so that $\mathscr{I}(\mathscr{C})\subset \mathscr{I}$. If i) holds, then $\mathscr{I}(\mathscr{C})$ contains every homogeneous invariant of positive degree, so that $\mathscr{I}\subset \mathscr{I}(\mathscr{C})\Rightarrow \mathscr{I}=\mathscr{I}(\mathscr{C})$. Thus i) \Rightarrow ii).

Suppose ii) holds. Choose in \mathscr{C} a minimal basis for \mathscr{I} . The proof of Chevalley's Theorem shows that this minimal basis consists of n homogeneous invariants $P_1, ..., P_n$ which are algebraically independent

$$\Leftrightarrow \frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} \neq 0.$$

According to Theorem 3.1, these degrees must be $d_1, ..., d_n$. Thus ii) \Rightarrow iii). Finally, the implication iii) \Rightarrow i) is contained in Theorem 3.13.

LEMMA 4.4. Let G be a finite reflection group acting on k^n . Let $I_1, ..., I_n$ be a basic set of homogeneous invariants of respective positive degrees $d_1, ..., d_n$ which are assumed distinct; i.e. $d_1 < d_2 < ... < d_n$. Let $P_1, ..., P_n$ be another set of homogeneous invariants of respective degrees $d_1, ..., d_n$. Thus

(4.17)
$$P_{i}(x) = F_{i}(I_{1}(x), \dots, I_{i-1}(x)) + c_{i}I_{i}(x)$$
$$= F_{i}(x) + c_{i}I_{i}(x), 1 \leq i \leq n$$

where $F_i(x)$ is homogeneous of degree m_i , with $F_1 = 0$, and c_i a constant. Then

(4.18)
$$\frac{\partial (P_1, \dots, P_n)}{\partial (x_1, \dots, x_n)} = c_1 \dots c_n \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)}$$

Proof. We have

$$\frac{\partial (P_1, \dots, P_n)}{\partial (x_1, \dots, x_n)} = \frac{\partial (F_1, \dots, F_n)}{\partial (I_1, \dots, I_n)} \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)}$$

The matrix
$$\left[\frac{\partial F_i}{\partial I_j}\right]$$
 is triangular and $\frac{\partial F_i}{\partial I_i} = c_i$, $1 \le i \le n$, so that $\frac{\partial (F_1, \dots, F_n)}{\partial (x_1, \dots, x_n)} = c_1 \dots c_n$.

THEOREM 4.6 (Flatto and Wiener [10]). i) Let S_y be space of continuous functions on the n-dimensional region \mathcal{R} satisfying the mean value property (4.14). $S_y = D \Pi$ iff $G \neq D_{2n}$, $2 \leqslant n < \infty$, and

$$\frac{\partial (P_{d_1}, \ldots, P_{d_n})}{\partial (x_1, \ldots, x_n)} \neq 0.$$

ii) For $G \neq D_{2n}$, $2 \leqslant n < \infty$, we have

(4.19)
$$\frac{\partial \left(P_{d_1}, \dots, P_{d_n}\right)}{\partial \left(x_1, \dots, x_n\right)} = J_1(y) \dots J_n(y) \Pi(x)$$

the J's being a basic set of homogeneous invariants for G. Hence

$$S_v = D \Pi \text{ iff } J_1(y) \dots J_n(y) \neq 0.$$

Proof. According to Theorem 4.5, S is the solution space of

$$(4.20) f \in C^{\infty} \text{ and } p(\partial) f = 0, x \in \mathcal{R} \text{ and } p \in \mathcal{P}_{y}.$$

where $\mathscr{P}_y = (P_1(x, y), ..., P_m(x, y), ...)$. It follows from Theorems 4.1, 4.2 that $S_y = D \Pi$ iff $\mathscr{P}_y = \mathscr{I}$. By Lemma 4.3, $\mathscr{P}_y = \mathscr{I}$ iff the degrees $d_1, ..., d_n$ are distinct and

$$\frac{\partial (P_{d_1}, \dots, P_{d_n})}{\partial (x_1, \dots, x_n)} \neq 0$$

An inspection of the table in section 3.3 reveals that the d_i 's are distinct except when $G = D_{2n}$, $2 \le n < \infty$, in which case two d_i 's equal 2n.

ii) For each *n*-tuple $a = (a_1, ..., a_n)$ of non-negative integers, let $J_a(x)$

$$=\frac{1}{\mid G\mid} \sum_{\sigma \in G} (\sigma x)^a$$
. We have

$$P_{m}(x, y) = \frac{1}{|G|} \sum_{\sigma \in G} (\sigma x, y)^{m} = \frac{1}{|G|^{2}} \sum_{\sigma_{1} \in G} \sum_{\sigma_{2} \in G} (\sigma_{1} x, \sigma_{2} y)^{m} = \frac{1}{|G|^{2}} \sum_{|a|=m} \sum_{\sigma_{1} \in G} \sum_{\sigma_{2} \in G} \frac{m!}{a!} (\sigma_{1} x)^{a} (\sigma_{2} y)^{a} = \sum_{|a|=m} \frac{m!}{a!} J_{a}(x) J_{a}(y)$$

Let $I_1, ..., I_n$ be a basic set of homogeneous invariants of respective degrees $d_1, ..., d_n$. Let $|a| = d_i$, $1 \le i \le n$. Then

$$(4.22) J_a(x) = F_a(I_1(x), \dots, I_{i-1}(x)) + c_a I_i(x) = F_a(x) + c_a I_i(x)$$

where $F_a(x)$ is homogeneous of degree d_i with $F_a(x) = 0$ for i = 1, and c_a is a constant. (4.21), (4.22) give

$$(4.23) P_{d_i}(x, y) = \sum_{|a|=d_i} \frac{d_i!}{a!} J_a(y) F_a(x) + J_i(y) I_i(x), \ 1 \le i \le n$$

where

(4.24)
$$J_i(y) = \sum_{|a|=d_i} \frac{d_i!}{a!} c_a J_a(y), \ 1 \leq i \leq n$$

(4.19) follows from (4.23) and Lemma 4.4. J_i is homogeneous of degree d_i . We show that $J_1, ..., J_n$ are algebraically independent and thus conclude from Lemma 4.3 that $J_1, ..., J_n$ form a basis for the invariants of G. Now the J'_a s form a basis for the invariants of G (see Noether's proof of Theorem 1.1). Hence, by Lemma 4.3, there exists $n J'_a$ s of respective degrees $d_1, ..., d_n$ which are algebraically independent. By Lemma 4.4, for each of these J'_a s, $c_a \neq 0$. (4.22), (4.24) give

$$(4.25) \quad J_i(y) = \sum_{|a|=d_i} \frac{d_i!}{a!} c_a F_a(y) + \left(\sum_{|a|=m_i} \frac{d_i}{a!} c_a^2\right) I_i(y), \ 1 \leqslant i \leqslant n$$

For each $1 \le i \le n$, there exists an a such that $|a| = d_i$ and $c_a \ne 0$, so that the n constants $\sum_{|n|=d_i} \frac{d_i}{a!} c_a^2$ are all $\ne 0$. It follows from (4.25) and Lemma 4.4, that $J_1, ..., J_n$ are algebraically independent.

The following theorem yields an algebraic characterization of the J'_i s.

THEOREM 4.7 [12]. $J_1(x) = c \sum_{i=1}^n x_i^2, c \neq 0$. For $2 \leqslant i \leqslant n$, $J_i(x)$ is determined up to a constant as the homogeneous invariant of degree d_i which satisfies the differential equations $J_k(\delta) J_i(x) = 0, 1 \leqslant k < i$.

Proof. $J_1(x)$ is a non-zero homogeneous invariant of degree 2 and must therefore be a non-zero multiple of $\sum_{i=1}^{n} x_i^2$. Let $2 \le i \le n$ and $1 \le k < d_i$. Let Q(x) be an arbitrary homogeneous invariant polynomial of degree k. We have

$$(4.26) Q(\partial_y) P_m(x, y) = Q(\partial_y) \left[\frac{1}{|G|} \sum_{\sigma \in G} (y, \sigma x)^m \right]$$
$$= m(m-1) \dots (m-k+1) P_{m-k}(x, y) Q(x)$$

From (4.23), we obtain

$$(4.27) Q(\partial_{y}) P_{d_{i}}(x, y)$$

$$= \sum_{|a|=d_{i}} \frac{d_{i}!}{a!} [Q(\partial) J_{a}(y)] F_{a}(x) + [Q(\partial) J_{i}(y)] I_{i}(x),$$

$$1 \leq i \leq n$$

so that

$$d_i(d_i-1) - (d_i-k+1) P_{d_i-k}(x, y) Q(x)$$

$$(4.28) = \sum_{|a|=d_i} \frac{d_i!}{a!} \left[Q(\partial) J_a(y) \right] F_a(x) + \left[Q(\partial) J_i(y) \right] I_i(x),$$

$$1 \leq i \leq n$$

Suppose that $Q(\delta)$ $J_i(y) \neq 0$. Choose y_0 so that $Q(\delta)$ $J_i(y) \neq 0$ at y_0 . Let $y = y_0$ in (4.28). The polynomial $P_{d_i-k}(x,y_0)$ has degree $< d_i$ and thus is a polynomial in $I_1, (x), ..., I_{i-1}(x)$. Each F_a is also a polynomial in $I_1, ..., I_{i-1}$. We conclude from (4.28) that $I_1, ..., I_i$ are algebraically dependent, a contradiction. Hence $Q(\delta)$ $J_k(y) = 0$, so that $J_k(\delta)$ $J_i(x) = 0$, $1 \leq k < i$.

The conditions of Theorem 4.7 determine J_i up to a constant. For let V_i = space of homogeneous invariants of degree d_i , W_i = space of homogeneous invariants of degree d_i spanned by the monomials in $I_1, ..., I_{i-1}$. Then dim V_i = dim W + 1. For any $J \in V_i$, the conditions $J_k(\delta)J(x) = 0$, $1 \le k < i$, are equivalent to $J \in W_i^{\perp}$. Since dim W_i^{\perp} = dim V_i - dim W_i = 1, we conclude that J_i is determined up to a constant.

COROLLARY. The manifold $\mathcal{M} = \{y \mid J_1(y) - - J_n(y) = 0\}$ contains real points $y \neq 0$. I.e. there exists $y \in \mathbb{R}^n$ such that $S \neq D \Pi$.

Proof. For $2 \le i \le n$, $J_1(\delta)J_i(x) = 0$. Since $J_1(x) = c \sum_{i=1}^n x_i^2$, $c \ne 0$, this means that $J_i(x)$ is harmonic. By the mean value property for harmonic functions, the average value of $J_i(y)$ on a sphere of radius $r > 0 = J_i(0) = 0$. Thus $J_i(y)$ must change sign on this sphere and a connectedness argument yields the existence of a $y \ne 0$ for which $J_i(y) = 0$.

In view of Theorem 4.6, we call \mathcal{M} the "exceptional manifold" for G and the non-zero vectors y of \mathcal{M} , the "exceptional directions" for G. A geometric description of \mathcal{M} is given in [24] for the groups H_2^n and A_3 . There remains the problem of describing the solution space S_y to the m.v.p. (4.14) in case y is an exceptional direction, as D Π is then a proper subspace of S_y . This seems to be a difficult problem. In [11], it is solved for the groups H_2^n , A_3 .