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By lemma 3.4 $P^*(x) = \Pi(x) i(x)$, where i is a homogeneous invariant. If $\deg i > 0$, then $P^* \in \mathcal{I} \Rightarrow P \in \mathcal{I}$. Otherwise $P^* = c \Pi$, c a constant. By assumption $P(\partial) \Pi = 0$, while $a(\partial) \Pi = 0$ for $a \in \mathcal{I}$. It follows that $P^*(\partial) \Pi = c(\Pi, \Pi) \Rightarrow c = 0$, so that $P \equiv 0 \pmod{\mathcal{I}}$.

2. MEAN VALUE PROPERTIES

We prove the equivalence of system (4.1) and a certain mean value property.

THEOREM 4.3 (Steinberg [21]). *Let $f(x) \in C$ in the n -dimensional region \mathcal{R} and let it satisfy the mean value property (m.v.p.)*

$$(4.6) \quad f(x) = \frac{1}{|G|} \sum_{\sigma \in G} f(x + \sigma y), \quad x \in \mathcal{R} \text{ and } \|y\| < \varepsilon_x,$$

where $\inf_{x \in K} \varepsilon_x > 0$ for any compact subset K of \mathcal{R} and $\|y\|^2 = \sum_{i=1}^n y_i^2$. This m.v.p. is equivalent to having $f \in C^\infty$ and satisfying (4.1). It follows from Theorem 4.2 that the space S of continuous solutions to (4.6) = $D \Pi$.

REMARK. The harmonic functions on \mathcal{R} are characterized as the continuous functions on \mathcal{R} satisfying the m.v.p. $f(x) = \int f(x+y) d\sigma(y)$, $x \in \mathcal{R}$ and $\|y\| < \varepsilon_x$, where $d\sigma(y)$ is the normalized Haar measure on the orthogonal group $O(n)$. (4.6) is just the G -analog of this m.v.p.

Proof of Theorem 4.3. Suppose first that $f(x)$ is C^∞ on \mathcal{R} and satisfies (4.6). Let $a(x)$ be any homogeneous invariant of positive degree. Apply the operator $a(\partial_y)$ to both sides of (4.6). In view of Lemma 4.1, we get

$$(4.7) \quad \begin{aligned} 0 &= a(\partial_y) f(x) = \frac{1}{|G|} \sum_{\sigma \in G} a(\partial_y) f(x + \sigma y) \\ &= \frac{1}{|G|} \sum_{\sigma \in G} [a(\partial_y) f(x + y)](\sigma y) \end{aligned}$$

Use $a(\partial_y) f(x+y) = a(\partial_x) f(x+y)$ and set $y = 0$. We obtain $a(\partial_x) f(x) = 0$, $x \in \mathcal{R}$ and a any homogeneous invariant of positive degree. Hence $a(\partial_x) f(x) = 0$, $x \in \mathcal{R}$ and $a \in \mathcal{I}$. Since $\sum_{i=1}^n x_i^2 \in \mathcal{I}$, we conclude in particular that $f(x)$ is harmonic on \mathcal{R} .

Suppose next that $f(x)$ is C on \mathcal{R} and satisfies (4.6). Let $\{\delta_k\}$ be a sequence of C^∞ functions on R^n such that $\int \delta_k(x) dx = 1$, support of $\delta_k = \left\{x \mid \|x\| \leq \frac{1}{k}\right\}$, $\delta_k(x) \geq 0$ for all x and k . Let

$$f_k(x) = \int f(x-y) \delta_k(y) dy = \int f(y) \delta_k(x-y) dy.$$

It is readily checked that for any compact subset S of \mathcal{R} , $f_k(x) \in C^\infty$ on $\text{Int } S$ (= interior of S) and satisfies (4.6) with \mathcal{R} replaced by $\text{Int } S$, provided k is sufficiently large, and $f_k \rightarrow f$ uniformly on S as $k \rightarrow \infty$. For k sufficiently large, f_k is harmonic on $\text{Int } S$. It follows from Harnack's Theorem ([15], p. 248) that $f(x)$ is harmonic on \mathcal{R} . Hence $f(x)$ is real analytic on \mathcal{R} ([15], p. 251) and so certainly C^∞ on \mathcal{R} .

Conversely let $f \in C^\infty$ on \mathcal{R} and $a(\partial)f = 0$, $x \in \mathcal{R}$ and $a \in \mathcal{I}$. Then f is harmonic and so real analytic on \mathcal{R} . Hence there exists $\varepsilon_x > 0$ such that

$$f(x+y) = \sum_{m=0}^{\infty} \frac{1}{m!} (\partial_x, y)^m f(x), x \in \mathcal{R}$$

and $\|y\| < \varepsilon_x$. It follows that

$$(4.8) \quad \frac{1}{|G|} \sum_{\sigma \in G} f(x + \sigma y) = \sum_{m=0}^{\infty} \frac{P_m(\partial_x, y)}{m!} f(x), x \in \mathcal{R}$$

and $\|y\| < \varepsilon_x$ where

$$(4.9) \quad P_m(x, y) = \frac{1}{|G|} \sum_{\sigma \in G} (x, \sigma y)^m = \frac{1}{|G|} \sum_{\sigma \in G} (\sigma x, y)^m.$$

From (4.9), we see that for fixed y , each $P_m(x, y)$ is a homogeneous invariant polynomial in x of degree m . It follows that $P_m(\partial_x, y)f(x) = 0$, $x \in \mathcal{R}$ and $m \leq 1$, and (4.8) reduces to (4.6).

The solution space to either (4.1) or (4.6) is the finite dimensional vector space $D\Pi$. The following result gives further information on $D\Pi$.

THEOREM 4.4 (Chevalley [4]). *Let $S_m =$ vector space of homogeneous polynomials of degree m in $D\Pi$, $0 \leq m < \infty$, so that $D\Pi = \sum_{m=0}^{\infty} \oplus S_m$. Let d_1, \dots, d_n be the degrees of the basic homogeneous invariants for G . Then*

$$(4.10) \quad \sum_{m=0}^{\infty} (\dim S_m) t^m = \prod_{i=1}^n \frac{1 - t^{d_i}}{1 - t}$$

and $\dim D\Pi = |G|$.

We prove first the preliminary

LEMMA 4.2. Let $R = k[x_1, \dots, x_n]$ = ring of polynomials in x_1, \dots, x_n with coefficients from k , k being any field of characteristic 0. Let G be a finite reflection group acting on k^n and \mathcal{I} the ideal generated by homogeneous invariants of positive degree. For any polynomial P , let \bar{P} be its residue class in the residue class ring R/\mathcal{I} . Suppose that P_1, \dots, P_s are homogeneous polynomials such that $\bar{P}_1, \dots, \bar{P}_s$ are linearly independent over R/\mathcal{I} (the latter is a vector space over k). Then P_1, \dots, P_s are linearly independent over $k(I)$, the field obtained by adjoining the set I of all invariant polynomials to k .

Proof. Suppose $\sum_{i=1}^s V_i P_i = 0$ where $V_i \in k(I)$, $1 \leq i \leq s$. We may suppose that the V_i 's are homogeneous and $[\deg V_i + \deg P_i]$ is the same for all i . Let I_1, \dots, I_n be a basic set of homogeneous invariants of positive degree. Let S_j , $0 \leq j < \infty$, be the different monomials in $I_1 \dots I_n$ arranged by increasing x -degree, with $s_0 = 1$. Let $V_i = \sum_{j=0}^{\infty} k_{ij} S_j$, $1 \leq i \leq s$, the k_{ij} 's being elements of k , and define k_{i0} to be 0. We have

$$(4.11) \quad \sum_{i=1}^s V_i P_i = \sum_{j=0}^{\infty} \left[\sum_{i=1}^s k_{ij} P_i \right] S_j = 0$$

Assume, as induction hypothesis, that $k_{ij} = 0$ for $j < l$. Thus $\sum_{j=l}^{\infty} \left[\sum_{i=1}^s k_{ij} P_i \right] S_j = 0$. $S_l \notin$ ideal generated by the S_j 's, $j > l$, as I_1, \dots, I_n are algebraically independent. It follows from Lemma 2.1 that $\sum_{i=1}^s k_{il} P_i \in \mathcal{I} \Leftrightarrow \sum_{i=1}^s k_{il} \bar{P}_i = 0 \Leftrightarrow k_{il} = 0, 1 \leq i \leq s$. Hence all $k_{ij} = 0$ and $V_i = 0, 1 \leq i \leq s$. I.e. P_1, \dots, P_s are linearly independent over $k(I)$.

We now return to the proof of Theorem 4.4. Let A_1, \dots, A_q be homogeneous polynomials such that $\bar{A}_1, \dots, \bar{A}_q$ form a basis for R/\mathcal{I} . By induction on the degree, we see that every polynomial P may be expressed as

$$(4.12) \quad P = \sum_{i=1}^q J_i A_i$$

where the J_i 's are invariant polynomials. Lemma 4.2 shows that this representation is unique. Let $R_m =$ set of homogeneous polynomials of degree m , $I_m = I \cap R_m$, $(R/\mathcal{I})_m =$ vector space spanned by those \bar{A}_i 's for which degree $A_i = m$. Let

$$p_R(t) = \sum_{n=0}^{\infty} (\dim R_m) t^m, \quad p_I(t) = \sum_{m=0}^{\infty} (\dim I_m) t^m,$$

$$p_{R/\mathcal{I}}(t) = \sum_{m=0}^{\infty} \dim (R/\mathcal{I})_m t^m.$$

In view of the uniqueness of the representation (4.12), we have

$$(4.13) \quad p_R(t) = p_I(t) p_{R/\mathcal{I}}(t)$$

Now

$$p_I(t) = \frac{1}{\prod_{i=1}^n (1 - t^{d_i})} \quad (\text{formula (2.5)})$$

while

$$p_{R/\mathcal{I}}(t) = \frac{1}{(1 - t)^n}$$

(as $\dim R_m = \binom{m+n-1}{m}$). By Fischer's Theorem R/\mathcal{I} may be identified with $D\Pi$, so that $p_{R/\mathcal{I}}(t) = \sum_{m=0}^{\infty} (\dim S_m) t^m$. Thus (4.13) becomes (4.10).

Set $t = 1$ in (4.10). The left side becomes $\sum_{m=0}^{\infty} \dim S_m = \dim D\Pi$. Since

$$\frac{1 - t^{d_i}}{1 - t} = 1 + t + \dots + t^{d_i-1} = d_i$$

at $t = 1$, the right side becomes $\prod_{i=1}^n d_i = |G|$ (by Theorem 2.2). Thus $\dim D\Pi = |G|$.

We now describe the solution space to (4.6) when we restrict the direction of y . For simplicity, we restrict ourselves to irreducible groups (the reducible case is discussed in [12]).

THEOREM 4.5. *Let $f(x) \in C$ in the n -dimensional region \mathcal{R} and satisfy the m.v.p.*

$$(4.14) \quad f(x) = \frac{1}{|G|} \sum_{\sigma \in G} f(x + t\sigma y), \quad x \in \mathcal{R} \text{ and } 0 < t < \varepsilon_x,$$

$\inf_{x \in K} \varepsilon_x > 0$ for any compact subset K of \mathcal{R} and y denoting a fixed vector $\neq 0$. This m.v.p. is equivalent to having $f \in C^\infty$ on \mathcal{R} and $P_m(\partial_x, y)f = 0, x \in \mathcal{R}$ and $1 \leq m < \infty, P_m$ being defined by (4.9).

Proof. Suppose first that $f \in C^\infty$ on \mathcal{R} and satisfies (4.14). Using the finite Taylor expansion for $f(x + t\sigma y)$, we get for each integer $N \geq 0$

$$(4.15) \quad 0 = \sum_{m=1}^N \left[\frac{P_m(\partial_x, y)f}{m!} \right] t^m + O(t^{N+1}) \text{ as } t \rightarrow 0.$$

Dividing by successive powers of t and letting $t \rightarrow 0$, we conclude $P_m(\partial_x, y)f = 0, x \in \mathcal{R}$ and $1 \leq m < \infty$. If $f \in C$, then we argue as in the proof of Theorem 4.3, introducing the functions f_k . For any compact subset S of \mathcal{R} and k sufficiently large, the f_k 's will be C^∞ on $\text{Int } S$ and satisfy there $P_m(\partial_x, y)f = 0, 1 \leq m < \infty$. $P_2(x, y)$ is a non-zero homogeneous invariant of degree 2. For irreducible G , there is up to a multiplicative constant, only one such invariant, namely $\sum_{i=1}^n x_i^2$. Thus

$$P_2(x, y) = c(y) \sum_{i=1}^n x_i^2, \text{ where } c(y) \neq 0 \text{ is a constant depending on } y.$$

Thus for k sufficiently large, $f_k(x)$ is harmonic on $\text{Int } S$. Since $f_k \rightarrow f$ uniformly on compact subsets of \mathcal{R} , $f(x)$ is harmonic on \mathcal{R} and hence certainly C^∞ on \mathcal{R} .

Conversely, let $P_m(\partial_x, y)f = 0, x \in \mathcal{R}$ and $1 \leq m < \infty$. Since $P_2(\partial_x, y)f = 0, f$ is harmonic and so real analytic on \mathcal{R} . It follows that there exists $\varepsilon_x > 0$ such that

$$(4.16) \quad \frac{1}{|G|} \sum_{\sigma \in G} f(x + t\sigma y) = \sum_{m=0}^{\infty} \left[\frac{P_m(\partial_x, y)f}{m!} \right] t^m, \quad x \in \mathcal{R}$$

and $0 < t < \varepsilon_x$.

Since $P_m(\partial_x, y)f = 0, x \in \mathcal{R}$ and $1 \leq m < \infty$, (4.16) reduces to (4.14).

We shall describe the solution space to $P_m(\partial_x, y)f = 0, 1 \leq m < \infty, y$ being a fixed vector $\neq 0$. We first prove some preliminary lemmas.

LEMMA 4.3. Let \mathcal{C} be a collection of homogeneous polynomials in $k[x_1, \dots, x_n]$ of positive degree, k being a field of characteristic 0. Let G be a finite reflection group acting on k^n . The following conditions are equivalent.

i) \mathcal{C} is a basis for the invariants of G

- ii) \mathcal{C} is a basis for the ideal \mathcal{I} generated by the homogeneous invariants of positive degree.
- iii) Let d_1, \dots, d_n be the degrees of the basic homogeneous invariants of G .

For each d_i there exists a polynomial $P_i \in \mathcal{C}$ of degree d_i such that

$$\frac{\partial (P_1, \dots, P_n)}{\partial (x_1, \dots, x_n)} \neq 0 .$$

Proof. Let $\mathcal{I}(\mathcal{C}) =$ ideal generated by \mathcal{C} , so that $\mathcal{I}(\mathcal{C}) \subset \mathcal{I}$. If i) holds, then $\mathcal{I}(\mathcal{C})$ contains every homogeneous invariant of positive degree, so that $\mathcal{I} \subset \mathcal{I}(\mathcal{C}) \Rightarrow \mathcal{I} = \mathcal{I}(\mathcal{C})$.

Thus i) \Rightarrow ii).

Suppose ii) holds. Choose in \mathcal{C} a minimal basis for \mathcal{I} . The proof of Chevalley's Theorem shows that this minimal basis consists of n homogeneous invariants P_1, \dots, P_n which are algebraically independent

$$\Leftrightarrow \frac{\partial (P_1, \dots, P_n)}{\partial (x_1, \dots, x_n)} \neq 0 .$$

According to Theorem 3.1, these degrees must be d_1, \dots, d_n . Thus ii) \Rightarrow iii).

Finally, the implication iii) \Rightarrow i) is contained in Theorem 3.13.

LEMMA 4.4. Let G be a finite reflection group acting on k^n . Let I_1, \dots, I_n be a basic set of homogeneous invariants of respective positive degrees d_1, \dots, d_n which are assumed distinct; i.e. $d_1 < d_2 < \dots < d_n$. Let P_1, \dots, P_n be another set of homogeneous invariants of respective degrees d_1, \dots, d_n . Thus

$$(4.17) \quad \begin{aligned} P_i(x) &= F_i(I_1(x), \dots, I_{i-1}(x)) + c_i I_i(x) \\ &= F_i(x) + c_i I_i(x), \quad 1 \leq i \leq n \end{aligned}$$

where $F_i(x)$ is homogeneous of degree m_i , with $F_1 = 0$, and c_i a constant. Then

$$(4.18) \quad \frac{\partial (P_1, \dots, P_n)}{\partial (x_1, \dots, x_n)} = c_1 \dots c_n \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)}$$

Proof. We have

$$\frac{\partial (P_1, \dots, P_n)}{\partial (x_1, \dots, x_n)} = \frac{\partial (F_1, \dots, F_n)}{\partial (I_1, \dots, I_n)} \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)}$$

The matrix $\left[\frac{\partial F_i}{\partial I_j} \right]$ is triangular and $\frac{\partial F_i}{\partial I_i} = c_i, 1 \leq i \leq n$, so that

$$\frac{\partial (F_1, \dots, F_n)}{\partial (x_1, \dots, x_n)} = c_1 \dots c_n.$$

THEOREM 4.6 (Flatto and Wiener [10]). i) Let S_y be space of continuous functions on the n -dimensional region \mathcal{R} satisfying the mean value property (4.14). $S_y = D \Pi$ iff $G \neq D_{2n}, 2 \leq n < \infty$, and

$$\frac{\partial (P_{d_1}, \dots, P_{d_n})}{\partial (x_1, \dots, x_n)} \neq 0.$$

ii) For $G \neq D_{2n}, 2 \leq n < \infty$, we have

$$(4.19) \quad \frac{\partial (P_{d_1}, \dots, P_{d_n})}{\partial (x_1, \dots, x_n)} = J_1(y) \dots J_n(y) \Pi(x)$$

the J 's being a basic set of homogeneous invariants for G . Hence

$$S_y = D \Pi \text{ iff } J_1(y) \dots J_n(y) \neq 0.$$

Proof. According to Theorem 4.5, S is the solution space of

$$(4.20) \quad f \in C^\infty \text{ and } p(\partial)f = 0, x \in \mathcal{R} \text{ and } p \in \mathcal{P}_y.$$

where $\mathcal{P}_y = (P_1(x, y), \dots, P_m(x, y), \dots)$. It follows from Theorems 4.1, 4.2 that $S_y = D \Pi$ iff $\mathcal{P}_y = \mathcal{I}$. By Lemma 4.3, $\mathcal{P}_y = \mathcal{I}$ iff the degrees d_1, \dots, d_n are distinct and

$$\frac{\partial (P_{d_1}, \dots, P_{d_n})}{\partial (x_1, \dots, x_n)} \neq 0$$

An inspection of the table in section 3.3 reveals that the d_i 's are distinct except when $G = D_{2n}, 2 \leq n < \infty$, in which case two d_i 's equal $2n$.

ii) For each n -tuple $a = (a_1, \dots, a_n)$ of non-negative integers, let $J_a(x)$

$$= \frac{1}{|G|} \sum_{\sigma \in G} (\sigma x)^a. \text{ We have}$$

$$(4.21) \quad \begin{aligned} P_m(x, y) &= \frac{1}{|G|} \sum_{\sigma \in G} (\sigma x, y)^m = \frac{1}{|G|^2} \sum_{\sigma_1 \in G} \sum_{\sigma_2 \in G} (\sigma_1 x, \sigma_2 y)^m = \\ &= \frac{1}{|G|^2} \sum_{|a|=m} \sum_{\sigma_1 \in G} \sum_{\sigma_2 \in G} \frac{m!}{a!} (\sigma_1 x)^a (\sigma_2 y)^a = \sum_{|a|=m} \frac{m!}{a!} J_a(x) J_a(y) \end{aligned}$$

Let I_1, \dots, I_n be a basic set of homogeneous invariants of respective degrees d_1, \dots, d_n . Let $|a| = d_i, 1 \leq i \leq n$. Then

$$(4.22) \quad J_a(x) = F_a(I_1(x), \dots, I_{i-1}(x)) + c_a I_i(x) = F_a(x) + c_a I_i(x)$$

where $F_a(x)$ is homogeneous of degree d_i with $F_a(x) = 0$ for $i = 1$, and c_a is a constant. (4.21), (4.22) give

$$(4.23) \quad P_{d_i}(x, y) = \sum_{|a|=d_i} \frac{d_i!}{a!} J_a(y) F_a(x) + J_i(y) I_i(x), \quad 1 \leq i \leq n$$

where

$$(4.24) \quad J_i(y) = \sum_{|a|=d_i} \frac{d_i!}{a!} c_a J_a(y), \quad 1 \leq i \leq n$$

(4.19) follows from (4.23) and Lemma 4.4. J_i is homogeneous of degree d_i . We show that J_1, \dots, J_n are algebraically independent and thus conclude from Lemma 4.3 that J_1, \dots, J_n form a basis for the invariants of G . Now the J'_a s form a basis for the invariants of G (see Noether's proof of Theorem 1.1). Hence, by Lemma 4.3, there exists $n J'_a$ s of respective degrees d_1, \dots, d_n which are algebraically independent. By Lemma 4.4, for each of these J'_a s, $c_a \neq 0$. (4.22), (4.24) give

$$(4.25) \quad J_i(y) = \sum_{|a|=d_i} \frac{d_i!}{a!} c_a F_a(y) + \left(\sum_{|a|=m_i} \frac{d_i}{a!} c_a^2 \right) I_i(y), \quad 1 \leq i \leq n$$

For each $1 \leq i \leq n$, there exists an a such that $|a| = d_i$ and $c_a \neq 0$, so that the n constants $\sum_{|n|=d_i} \frac{d_i}{a!} c_a^2$ are all $\neq 0$. It follows from (4.25) and Lemma 4.4, that J_1, \dots, J_n are algebraically independent.

The following theorem yields an algebraic characterization of the J_i 's.

THEOREM 4.7 [12]. $J_1(x) = c \sum_{i=1}^n x_i^2, c \neq 0$. For $2 \leq i \leq n$, $J_i(x)$ is determined up to a constant as the homogeneous invariant of degree d_i which satisfies the differential equations $J_k(\partial) J_i(x) = 0, 1 \leq k < i$.

Proof. $J_1(x)$ is a non-zero homogeneous invariant of degree 2 and must therefore be a non-zero multiple of $\sum_{i=1}^n x_i^2$. Let $2 \leq i \leq n$ and $1 \leq k < d_i$. Let $Q(x)$ be an arbitrary homogeneous invariant polynomial of degree k . We have

$$(4.26) \quad \begin{aligned} Q(\partial_y) P_m(x, y) &= Q(\partial_y) \left[\frac{1}{|G|} \sum_{\sigma \in G} (y, \sigma x)^m \right] \\ &= m(m-1) \dots (m-k+1) P_{m-k}(x, y) Q(x) \end{aligned}$$

From (4.23), we obtain

$$(4.27) \quad Q(\partial_y) P_{d_i}(x, y) \\ = \sum_{|a|=d_i} \frac{d_i!}{a!} [Q(\partial) J_a(y)] F_a(x) + [Q(\partial) J_i(y)] I_i(x), \\ 1 \leq i \leq n$$

so that

$$(4.28) \quad d_i(d_i - 1) \dots (d_i - k + 1) P_{d_i - k}(x, y) Q(x) \\ = \sum_{|a|=d_i} \frac{d_i!}{a!} [Q(\partial) J_a(y)] F_a(x) + [Q(\partial) J_i(y)] I_i(x), \\ 1 \leq i \leq n$$

Suppose that $Q(\partial) J_i(y) \neq 0$. Choose y_0 so that $Q(\partial) J_i(y) \neq 0$ at y_0 . Let $y = y_0$ in (4.28). The polynomial $P_{d_i - k}(x, y_0)$ has degree $< d_i$ and thus is a polynomial in $I_1(x), \dots, I_{i-1}(x)$. Each F_a is also a polynomial in I_1, \dots, I_{i-1} . We conclude from (4.28) that I_1, \dots, I_i are algebraically dependent, a contradiction. Hence $Q(\partial) J_k(y) = 0$, so that $J_k(\partial) J_i(x) = 0, 1 \leq k < i$.

The conditions of Theorem 4.7 determine J_i up to a constant. For let $V_i =$ space of homogeneous invariants of degree $d_i, W_i =$ space of homogeneous invariants of degree d_i spanned by the monomials in I_1, \dots, I_{i-1} . Then $\dim V_i = \dim W + 1$. For any $J \in V_i$, the conditions $J_k(\partial) J(x) = 0, 1 \leq k < i$, are equivalent to $J \in W_i^\perp$. Since $\dim W_i^\perp = \dim V_i - \dim W_i = 1$, we conclude that J_i is determined up to a constant.

COROLLARY. The manifold $\mathcal{M} = \{y \mid J_1(y) \dots J_n(y) = 0\}$ contains real points $y \neq 0$. I.e. there exists $y \in R^n$ such that $S \neq D\Pi$.

Proof. For $2 \leq i \leq n, J_1(\partial) J_i(x) = 0$. Since $J_1(x) = c \sum_{i=1}^n x_i^2, c \neq 0$, this means that $J_i(x)$ is harmonic. By the mean value property for harmonic functions, the average value of $J_i(y)$ on a sphere of radius $r > 0 = J_i(0) = 0$. Thus $J_i(y)$ must change sign on this sphere and a connectedness argument yields the existence of a $y \neq 0$ for which $J_i(y) = 0$.

In view of Theorem 4.6, we call \mathcal{M} the "exceptional manifold" for G and the non-zero vectors y of \mathcal{M} , the "exceptional directions" for G . A geometric description of \mathcal{M} is given in [24] for the groups H_2^n and A_3 . There remains the problem of describing the solution space S_y to the m.v.p. (4.14) in case y is an exceptional direction, as $D\Pi$ is then a proper subspace of S_y . This seems to be a difficult problem. In [11], it is solved for the groups H_2^n, A_3 .