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By lemma 3.4 $P^*(x) = \Pi(x)\, i(x)$, where i is a homogeneous invariant. If $\deg i > 0$, then $P^* \in \mathscr{I} \Rightarrow P \in \mathscr{I}$. Otherwise $P^* = c\,\Pi$, c a constant. By assumption $P(\partial)\,\Pi = 0$, while $a(\partial)\,\Pi = 0$ for $a \in \mathscr{I}$. It follows that $P^*(\partial)\,\Pi = c\,(\Pi, \Pi) \Rightarrow c = 0$, so that $P \equiv 0 \,(\mathrm{mod}\; \mathscr{I})$.

2. Mean Value Properties

We prove the equivalence of system (4.1) and a certain mean value property.

Theorem 4.3 (Steinberg [21]). *Let $f(x) \in C$ in the n-dimensional region $\mathscr{R}$ and let it satisfy the mean value property (m.v.p.)*

$$(4.6) \qquad f(x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} f(x + \sigma y), \; x \in \mathscr{R} \text{ and } \| y \| < \varepsilon_x ,$$

where $\inf_{x \in K} \varepsilon_x > 0$ for any compact subset K of $\mathscr{R}$ and $\|y\|^2 = \sum_{i=1}^{n} y_i^2$. This m.v.p. is equivalent to having $f \in C^\infty$ and satisfying (4.1). It follows from Theorem 4.2 that the space S of continuous solutions to (4.6) $= D\,\Pi$.

Remark. The harmonic functions on $\mathscr{R}$ are characterized as the continuous functions on $\mathscr{R}$ satisfying the m.v.p. $f(x) = \int f(x+y)\, d\sigma(y)$, $x \in \mathscr{R}$ and $\| y \| < \varepsilon_x$, where $d\sigma(y)$ is the normalized Haar measure on the orthogonal group $O(n)$. (4.6) is just the G-analog of this m.v.p.

Proof of Theorem 4.3. Suppose first that $f(x)$ is C^∞ on $\mathscr{R}$ and satisfies (4.6). Let $a(x)$ be any homogeneous invariant of positive degree. Apply the operator $a(\partial_y)$ to both sides of (4.6). In view of Lemma 4.1, we get

$$(4.7) \qquad 0 = a(\partial_y) f(x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} a(\partial_y) f(x + \sigma y)$$

$$= \frac{1}{|G|} \sum_{\sigma \varepsilon G} \left[a(\partial_y) f(x + y) \right] (\sigma y)$$

Use $a(\partial_y) f(x+y) = a(\partial_x) f(x+y)$ and set $y = 0$. We obtain $a(\partial_x) f(x) = 0$, $x \in \mathscr{R}$ and a any homogeneous invariant of positive degree. Hence $a(\partial_x) f(x) = 0$, $x \in \mathscr{R}$ and $a \in \mathscr{I}$. Since $\sum_{i=1}^{n} x_i^2 \in \mathscr{I}$, we conclude in particular that $f(x)$ is harmonic on $\mathscr{R}$.

Suppose next that $f(x)$ is C on $\mathscr{R}$ and satisfies (4.6). Let $\{\delta_k\}$ be a sequence of C^∞ functions on R^n such that $\int \delta_k(x)\,dx = 1$, support of $\delta_k = \left\{x \mid \|x\| \leqslant \frac{1}{k}\right\}$, $\delta_k(x) \geqslant 0$ for all x and k. Let

$$f_k(x) = \int f(x-y)\,\delta_k(y)\,dy = \int f(y)\,\delta_k(x-y)\,dy\,.$$

It is readily checked that for any compact subset S of $\mathscr{R}$, $f_k(x) \in C^\infty$ on Int S (= interior of S) and satisfies (4.6) with $\mathscr{R}$ replaced by Int S, provided k is sufficiently large, and $f_k \to f$ uniformly on S as $k \to \infty$. For k sufficiently large, f_k is harmonic on Int S. It follows from Harnack's Theorem ([15], p. 248) that $f(x)$ is harmonic on $\mathscr{R}$. Hence $f(x)$ is real analytic on $\mathscr{R}$ ([15], p. 251) and so certainly C^∞ on $\mathscr{R}$.

Conversely let $f \in C^\infty$ on $\mathscr{R}$ and $a(\partial) f = 0$, $x \in \mathscr{R}$ and $a \in \mathscr{I}$. Then f is harmonic and so real analytic on $\mathscr{R}$. Hence there exists $\varepsilon_x > 0$ such that

$$f(x+y) = \sum_{m=0}^{\infty} \frac{1}{m!}(\partial_x, y)^m f(x), \; x \in \mathscr{R}$$

and $\|y\| < \varepsilon_x$. It follows that

$$\frac{1}{|G|} \sum_{\sigma \varepsilon G} f(x+\sigma y) = \sum_{m=0}^{\infty} \frac{P_m(\partial_x, y)}{m!} f(x), \quad x \in \mathscr{R} \tag{4.8}$$

and $\|y\| < \varepsilon_x$ where

$$P_m(x, y) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} (x, \sigma y)^m = \frac{1}{|G|} \sum_{\sigma \varepsilon G} (\sigma x, y)^m. \tag{4.9}$$

From (4.9), we see that for fixed y, each $P_m(x, y)$ is a homogeneous invariant polynomial in x of degree m. It follows that $P_m(\partial_x, y) f(x) = 0$, $x \in \mathscr{R}$ and $m \leqslant 1$, and (4.8) reduces to (4.6).

The solution space to either (4.1) or (4.6) is the finite dimensional vector space $D\,\Pi$. The following result gives further information on $D\,\Pi$.

Theorem 4.4 (Chevalley [4]). *Let* S_m = *vector space of homogeneous polynomials of degree* m *in* $D\,\Pi$, $0 \leqslant m < \infty$, *so that* $D\,\Pi = \sum_{m=0}^{\infty} \oplus S_m$. *Let* $d_1, \ldots, d_n$ *be the degrees of the basic homogeneous invariants for* G. *Then*

$$(4.10) \qquad \sum_{m=0}^{\infty} (\dim S_m)\, t^m = \prod_{i=1}^{n} \frac{1-t^{d_i}}{1-t}$$

and $\dim D\Pi = |G|$.

We prove first the preliminary

LEMMA 4.2. Let $R = k[x_1, ..., x_n]$ = ring of polynomials in $x_1, ..., x_n$ with coefficients from k, k being any field of characteristic 0. Let G be a finite reflection group acting on k^n and $\mathscr{I}$ the ideal generated by homogeneous invariants of positive degree. For any polynomial P, let $\bar{P}$ be its residue class in the residue class ring $R/\mathscr{I}$. Suppose that $P_1, ..., P_s$ are homogeneous polynomials such that $\bar{P}_1, ..., \bar{P}_s$ are linearly independent over $R/\mathscr{I}$ (the latter is a vector space over k). Then $P_1, ..., P_s$ are linearly independent over $k(I)$, the field obtained by adjoining the set I of all invariant polynomials to k.

Proof. Suppose $\sum_{i=1}^{s} V_i P_i = 0$ where $V_i \in k(I)$, $1 \leqslant i \leqslant s$. We may suppose that the V_i's are homogeneous and [$\deg V_i + \deg P_i$] is the same for all i. Let $I_1, ..., I_n$ be a basic set of homogeneous invariants of positive degree. Let S_j, $0 \leqslant j < \infty$, be the different monomials in $I_1 \dots I_n$ arranged by increasing x-degree, with $s_0 = 1$. Let $V_i = \sum_{j=0}^{\infty} k_{ij} S_j$, $1 \leqslant i \leqslant s$, the k_{ij}'s being elements of k, and define k_{i0} to be 0. We have

$$(4.11) \qquad \sum_{i=1}^{s} V_i P_i = \sum_{j=0}^{\infty} \Big[\sum_{i=1}^{s} k_{ij} P_i \Big] S_j = 0$$

Assume, as induction hypothesis, that $k_{ij} = 0$ for $j < l$. Thus $\sum_{j=l}^{\infty} [\sum_{i=1}^{s} k_{ij} P_i] S_j = 0$. $S_l \notin$ ideal generated by the S_j's, $j > l$, as $I_1, ..., I_n$ are algebraically independent. It follows from Lemma 2.1 that $\sum_{i=1}^{s} k_{il} P_i \in \mathscr{I} \Leftrightarrow \sum_{i=1}^{s} k_{il} \bar{P}_i = 0 \Leftrightarrow k_{il} = 0$, $1 \leqslant i \leqslant s$. Hence all $k_{ij} = 0$ and $V_i = 0$, $1 \leqslant i \leqslant s$. I.e. $P_1, ..., P_s$ are linearly independent over $k(I)$.

We now return to the proof of Theorem 4.4. Let $A_1, ..., A_q$ be homogeneous polynomials such that $\bar{A}_1, ..., \bar{A}_q$ form a basis for $R/\mathscr{I}$. By induction on the degree, we see that every polynomial P may be expressed as

$$(4.12) \qquad P = \sum_{i=1}^{q} J_i A_i$$

where the J_i's are invariant polynomials. Lemma 4.2 shows that this representation is unique. Let R_m = set of homogeneous polynomials of degree m, $I_m = I \cap R_m$, $(R/\mathcal{I})_m$ = vector space spanned by those $\bar{A}_i'$s for which degree $A_i = m$. Let

$$\mathfrak{p}_R(t) = \sum_{n=0}^{\infty} (\dim R_m)\, t^m, \quad \mathfrak{p}_I(t) = \sum_{m=0}^{\infty} (\dim I_m)\, t^m,$$

$$\mathfrak{p}_{R\mathcal{I}}(t) = \sum_{m=0}^{\infty} \dim (R/\mathcal{I})_m\, t^m.$$

In view of the uniqueness of the representation (4.12), we have

$$\mathfrak{p}_R(t) = \mathfrak{p}_I(t)\, \mathfrak{p}_{R/\mathcal{I}}(t) \tag{4.13}$$

Now

$$\mathfrak{p}_I(t) = \frac{1}{\prod_{i=1}^{n} (1 - t^{d_i})} \qquad \text{(formula (2.5))}$$

while

$$\mathfrak{p}_R(t) = \frac{1}{(1-t)^n}$$

(as $\dim R_m = \binom{m+n-1}{m}$). By Fischer's Theorem $R/\mathcal{I}$ may be identified with $D\,\Pi$, so that $\mathfrak{p}_{R/\mathcal{I}}(t) = \sum_{m=0}^{\infty} (\dim S_m)\, t^m$. Thus (4.13) becomes (4.10).

Set $t = 1$ in (4.10). The left side becomes $\sum_{m=0}^{\infty} \dim S_m = \dim D\,\Pi$. Since

$$\frac{1 - t^{d_i}}{1 - t} = 1 + t + \ldots + t^{d_i - 1} = d_i$$

at $t = 1$, the right side becomes $\prod_{i=1}^{n} d_i = |G|$ (by Theorem 2.2). Thus $\dim D\,\Pi = |G|$.

We now describe the solution space to (4.6) when we restrict the direction of y. For simplicity, we restrict ourselves to irreducible groups (the reducible case is discussed in [12]).

THEOREM 4.5. *Let $f(x) \in C$ in the n-dimensional region $\mathcal{R}$ and satisfy the m.v.p.*

$$f(x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} f(x + t\sigma y), \ x \in \mathcal{R} \text{ and } 0 < t < \varepsilon_x, \tag{4.14}$$

$\inf_{x \in K} \varepsilon_x > 0$ for any compact subset K of $\mathscr{R}$ and y denoting a fixed vector $\neq 0$. This m.v.p. is equivalent to having $f \in C^{\infty}$ on $\mathscr{R}$ and $P_m(\partial_x, y) f = 0$, $x \in \mathscr{R}$ and $1 \leqslant m < \infty$, P_m being defined by (4.9).

Proof. Suppose first that $f \in C^{\infty}$ on $\mathscr{R}$ and satisfies (4.14). Using the finite Taylor expansion for $f(x + t\sigma y)$, we get for each integer $N \geqslant 0$

$$(4.15) \qquad 0 = \sum_{m=1}^{N} \left[\frac{P_m(\partial_x, y) f}{m\,!} \right] t^m + O(t^{N+1}) \text{ as } t \to 0\,.$$

Dividing by successive powers of t and letting $t \to 0$, we conclude $P_m(\partial_x, y) f = 0$, $x \in \mathscr{R}$ and $1 \leqslant m < \infty$. If $f \in C$, then we argue as in the proof of Theorem 4.3, introducing the functions f_k. For any compact subset S of $\mathscr{R}$ and k sufficiently large, the f_k's will be C^{∞} on Int S and satisfy there $P_m(\partial_x, y) f = 0$, $1 \leqslant m < \infty$. $P_2(x, y)$ is a non-zero homogeneous invariant of degree 2. For irreducible G, there is up to a multiplicative constant, only one such invariant, namely $\sum_{i=1}^{n} x_i^2$. Thus $P_2(x, y) = c(y) \sum_{i=1}^{n} x_i^2$, where $c(y) \neq 0$ is a constant depending on y. Thus for k sufficiently large, $f_k(x)$ is harmonic on Int S. Since $f_k \to f$ uniformly on compact subsets of $\mathscr{R}$, $f(x)$ is harmonic on $\mathscr{R}$ and hence certainly C^{∞} on $\mathscr{R}$.

Conversely, let $P_m(\partial_x, y) f = 0$, $x \in \mathscr{R}$ and $1 \leqslant m < \infty$. Since $P_2(\partial_x, y) f = 0$, f is harmonic and so real analytic on $\mathscr{R}$. It follows that there exists $\varepsilon_x > 0$ such that

$$(4.16) \qquad \frac{1}{|G|} \sum_{\sigma \varepsilon G} f(x + t\sigma y) = \sum_{m=0}^{\infty} \left[\frac{P_m(\partial_x, y) f}{m\,!} \right] t^m, \; x \in \mathscr{R}$$

and $0 < t < \varepsilon_x$.

Since $P_m(\partial_x, y) f = 0$, $x \in \mathscr{R}$ and $1 \leqslant m < \infty$, (4.16) reduces to (4.14).

We shall describe the solution space to $P_m(\partial_x, y) f = 0, 1 \leqslant m < \infty$, y being a fixed vector $\neq 0$. We first prove some preliminary lemmas.

Lemma 4.3. Let $\mathscr{C}$ be a collection of homogeneous polynomials in $k[x_1 \ldots, x_n]$ of positive degree, k being a field of characteristic 0. Let G be a finite reflection group acting on k^n. The following conditions are equivalent.

i) $\mathscr{C}$ is a basis for the invariants of G

ii) $\mathscr{C}$ is a basis for the ideal $\mathscr{I}$ generated by the homogeneous invariants of positive degree.

iii) Let $d_1, ..., d_n$ be the degrees of the basic homogeneous invariants of G.

For each d_i there exists a polynomial $P_i \in \mathscr{C}$ of degree d_i such that

$$\frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} \neq 0 .$$

Proof. Let $\mathscr{I}(\mathscr{C})$ = ideal generated by $\mathscr{C}$, so that $\mathscr{I}(\mathscr{C}) \subset \mathscr{I}$. If i) holds, then $\mathscr{I}(\mathscr{C})$ contains every homogeneous invariant of positive degree, so that $\mathscr{I} \subset \mathscr{I}(\mathscr{C}) \Rightarrow \mathscr{I} = \mathscr{I}(\mathscr{C})$.
Thus i) $\Rightarrow$ ii).

Suppose ii) holds. Choose in $\mathscr{C}$ a minimal basis for $\mathscr{I}$. The proof of Chevalley's Theorem shows that this minimal basis consists of n homogeneous invariants $P_1, ..., P_n$ which are algebraically independent

$$\Leftrightarrow \frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} \neq 0 .$$

According to Theorem 3.1, these degrees must be $d_1, ..., d_n$. Thus ii) $\Rightarrow$ iii).

Finally, the implication iii) $\Rightarrow$ i) is contained in Theorem 3.13.

LEMMA 4.4. Let G be a finite reflection group acting on k^n. Let $I_1, ..., I_n$ be a basic set of homogeneous invariants of respective positive degrees $d_1, ..., d_n$ which are assumed distinct; i.e. $d_1 < d_2 < ... < d_n$. Let $P_1, ..., P_n$ be another set of homogeneous invariants of respective degrees $d_1, ..., d_n$. Thus

$$\begin{aligned} P_i(x) &= F_i(I_1(x), \ldots, I_{i-1}(x)) + c_i I_i(x) \\ &= F_i(x) + c_i I_i(x), \ 1 \leqslant i \leqslant n \end{aligned} \tag{4.17}$$

where $F_i(x)$ is homogeneous of degree m_i, with $F_1 = 0$, and c_i a constant. Then

$$\frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} = c_1 \ldots c_n \frac{\partial (I_1, \ldots, I_n)}{\partial (x_1, \ldots, x_n)} \tag{4.18}$$

Proof. We have

$$\frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} = \frac{\partial (F_1, \ldots, F_n)}{\partial (I_1, \ldots, I_n)} \frac{\partial (I_1, \ldots, I_n)}{\partial (x_1, \ldots, x_n)}$$

The matrix $\left[\dfrac{\partial F_i}{\partial I_j}\right]$ is triangular and $\dfrac{\partial F_i}{\partial I_i} = c_i,\ 1 \leqslant i \leqslant n$, so that

$$\frac{\partial (F_1, \dots, F_n)}{\partial (x_1, \dots, x_n)} = c_1 \dots c_n.$$

THEOREM 4.6 (Flatto and Wiener [10]). i) *Let* S_y *be space of continuous functions on the* n*-dimensional region* $\mathcal{R}$ *satisfying the mean value property (4.14).* $S_y = D\,\Pi$ *iff* $G \neq D_{2n},\ 2 \leqslant n < \infty$, *and*

$$\frac{\partial (P_{d_1}, \dots, P_{d_n})}{\partial (x_1, \dots, x_n)} \neq 0 .$$

ii) *For* $G \neq D_{2n},\ 2 \leqslant n < \infty$, *we have*

$$\frac{\partial (P_{d_1}, \dots, P_{d_n})}{\partial (x_1, \dots, x_n)} = J_1(y) \dots J_n(y)\, \Pi(x) \tag{4.19}$$

the J'*s being a basic set of homogeneous invariants for* G. *Hence*

$$S_y = D\,\Pi \text{ iff } J_1(y) \dots J_n(y) \neq 0 .$$

Proof. According to Theorem 4.5, S is the solution space of

$$f \in C^\infty \text{ and } p(\partial) f = 0,\ x \in \mathcal{R} \text{ and } p \in \mathcal{P}_y. \tag{4.20}$$

where $\mathcal{P}_y = (P_1(x, y), \dots, P_m(x, y), \dots)$. It follows from Theorems 4.1, 4.2 that $S_y = D\,\Pi$ iff $\mathcal{P}_y = \mathcal{I}$. By Lemma 4.3, $\mathcal{P}_y = \mathcal{I}$ iff the degrees $d_1, \dots, d_n$ are distinct and

$$\frac{\partial (P_{d_1}, \dots, P_{d_n})}{\partial (x_1, \dots, x_n)} \neq 0$$

An inspection of the table in section 3.3 reveals that the d_i's are distinct except when $G = D_{2n},\ 2 \leqslant n < \infty$, in which case two d_i's equal $2n$.
ii) For each n-tuple $a = (a_1, \dots, a_n)$ of non-negative integers, let $J_a(x) = \dfrac{1}{|G|} \sum\limits_{\sigma \in G} (\sigma x)^a$. We have

$$P_m(x, y) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} (\sigma x, y)^m = \frac{1}{|G|^2} \sum_{\sigma_1 \varepsilon G} \sum_{\sigma_2 \varepsilon G} (\sigma_1 x, \sigma_2 y)^m =$$

$$\frac{1}{|G|^2} \sum_{|a| = m} \sum_{\sigma_1 \varepsilon G} \sum_{\sigma_2 \varepsilon G} \frac{m!}{a!} (\sigma_1 x)^a (\sigma_2 y)^a = \sum_{|a| = m} \frac{m!}{a!} J_a(x)\, J_a(y) \tag{4.21}$$

Let $I_1, \dots, I_n$ be a basic set of homogeneous invariants of respective degrees $d_1, \dots, d_n$. Let $|a| = d_i,\ 1 \leqslant i \leqslant n$. Then

$$J_a(x) = F_a(I_1(x), \ldots, I_{i-1}(x)) + c_a I_i(x) = F_a(x) + c_a I_i(x) \tag{4.22}$$

where $F_a(x)$ is homogeneous of degree d_i with $F_a(x) = 0$ for $i = 1$, and c_a is a constant. (4.21), (4.22) give

$$P_{d_i}(x, y) = \sum_{|a|=d_i} \frac{d_i!}{a!} J_a(y) F_a(x) + J_i(y) I_i(x), \; 1 \leqslant i \leqslant n \tag{4.23}$$

where

$$J_i(y) = \sum_{|a|=d_i} \frac{d_i!}{a!} c_a J_a(y), \; 1 \leqslant i \leqslant n \tag{4.24}$$

(4.19) follows from (4.23) and Lemma 4.4. J_i is homogeneous of degree d_i. We show that $J_1, \ldots, J_n$ are algebraically independent and thus conclude from Lemma 4.3 that $J_1, \ldots, J_n$ form a basis for the invariants of G. Now the J_a's form a basis for the invariants of G (see Noether's proof of Theorem 1.1). Hence, by Lemma 4.3, there exists $n\, J_a'$s of respective degrees $d_1, \ldots, d_n$ which are algebraically independent. By Lemma 4.4, for each of these J_a's, $c_a \neq 0$. (4.22), (4.24) give

$$J_i(y) = \sum_{|a|=d_i} \frac{d_i!}{a!} c_a F_a(y) + \Big(\sum_{|a|=m_i} \frac{d_i}{a!} c_a^2 \Big) I_i(y), \; 1 \leqslant i \leqslant n \tag{4.25}$$

For each $1 \leqslant i \leqslant n$, there exists an a such that $|a| = d_i$ and $c_a \neq 0$, so that the n constants $\sum_{|n|=d_i} \frac{d_i}{a!} c_a^2$ are all $\neq 0$. It follows from (4.25) and Lemma 4.4, that $J_1, \ldots, J_n$ are algebraically independent.

The following theorem yields an algebraic characterization of the J_i's.

THEOREM 4.7 [12]. $J_1(x) = c \sum_{i=1}^{n} x_i^2$, $c \neq 0$. *For* $2 \leqslant i \leqslant n$, $J_i(x)$ *is determined up to a constant as the homogeneous invariant of degree* d_i *which satisfies the differential equations* $J_k(\partial) J_i(x) = 0$, $1 \leqslant k < i$.

Proof. $J_1(x)$ is a non-zero homogeneous invariant of degree 2 and must therefore be a non-zero multiple of $\sum_{i=1}^{n} x_i^2$. Let $2 \leqslant i \leqslant n$ and $1 \leqslant k < d_i$. Let $Q(x)$ be an arbitrary homogeneous invariant polynomial of degree k. We have

$$\begin{aligned} Q(\partial_y) P_m(x, y) &= Q(\partial_y) \Big[\frac{1}{|G|} \sum_{\sigma \varepsilon G} (y, \sigma x)^m \Big] \\ &= m(m-1) \ldots (m-k+1) P_{m-k}(x, y) \; Q(x) \end{aligned} \tag{4.26}$$

From (4.23), we obtain

$$(4.27)\qquad Q(\partial_y) P_{d_i}(x, y) = \sum_{|a|=d_i} \frac{d_i!}{a!} [Q(\partial) J_a(y)] F_a(x) + [Q(\partial) J_i(y)] I_i(x), \quad 1 \leqslant i \leqslant n$$

so that

$$(4.28)\qquad d_i(d_i-1) \cdots (d_i-k+1) P_{d_i-k}(x, y) Q(x) = \sum_{|a|=d_i} \frac{d_i!}{a!} [Q(\partial) J_a(y)] F_a(x) + [Q(\partial) J_i(y)] I_i(x), \quad 1 \leqslant i \leqslant n$$

Suppose that $Q(\partial) J_i(y) \neq 0$. Choose y_0 so that $Q(\partial) J_i(y) \neq 0$ at y_0. Let $y = y_0$ in (4.28). The polynomial $P_{d_i-k}(x, y_0)$ has degree $< d_i$ and thus is a polynomial in $I_1, (x), \ldots, I_{i-1}(x)$. Each F_a is also a polynomial in $I_1, \ldots, I_{i-1}$. We conclude from (4.28) that $I_1, \ldots, I_i$ are algebraically dependent, a contradiction. Hence $Q(\partial) J_k(y) = 0$, so that $J_k(\partial) J_i(x) = 0$, $1 \leqslant k < i$.

The conditions of Theorem 4.7 determine J_i up to a constant. For let V_i = space of homogeneous invariants of degree d_i, W_i = space of homogeneous invariants of degree d_i spanned by the monomials in $I_1, \ldots, I_{i-1}$. Then $\dim V_i = \dim W + 1$. For any $J \in V_i$, the conditions $J_k(\partial) J(x) = 0$, $1 \leqslant k < i$, are equivalent to $J \in W_i^{\perp}$. Since $\dim W_i^{\perp} = \dim V_i - \dim W_i = 1$, we conclude that J_i is determined up to a constant.

COROLLARY. The manifold $\mathscr{M} = \{y \mid J_1(y) \cdots J_n(y) = 0\}$ contains real points $y \neq 0$. I.e. there exists $y \in R^n$ such that $S \neq D\Pi$.

Proof. For $2 \leqslant i \leqslant n$, $J_1(\partial) J_i(x) = 0$. Since $J_1(x) = c \sum_{i=1}^{n} x_i^2$, $c \neq 0$, this means that $J_i(x)$ is harmonic. By the mean value property for harmonic functions, the average value of $J_i(y)$ on a sphere of radius $r > 0 = J_i(0) = 0$. Thus $J_i(y)$ must change sign on this sphere and a connectedness argument yields the existence of a $y \neq 0$ for which $J_i(y) = 0$.

In view of Theorem 4.6, we call $\mathscr{M}$ the "exceptional manifold" for G and the non-zero vectors y of $\mathscr{M}$, the "exceptional directions" for G. A geometric description of $\mathscr{M}$ is given in [24] for the groups H_2^n and A_3. There remains the problem of describing the solution space S_y to the m.v.p. (4.14) in case y is an exceptional direction, as $D\Pi$ is then a proper subspace of S_y. This seems to be a difficult problem. In [11], it is solved for the groups H_2^n, A_3.