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CHAPTER IV

PARTIAL DIFFERENTIAL EQUATIONS AND MEAN VALUE PROPERTIES

1. Invariant partial differential equations

We study in the present chapter a certain system of partial differential equations invariant under a finite reflection group G and related mean value properties. We assume throughout that the underlying field k is real (this permits us to introduce the methods of analysis) and that G is orthogonal, which can always be achieved after a linear change of variables. We rely on the invariant theory of the previous chapters to establish the forthcoming results. Conversely, we shall see that the problems studied in this chapter lead to a natural set of basic invariants for G. In the sequel, let R denote the ring of polynomials $k[x_1, \ldots, x_n]$. For any polynomial $p(x)$, $p(\partial)$ denotes the partial differential operator obtained by replacing $x = (x_1, \ldots, x_n)$ by the symbol

$$\partial = \partial_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right).$$

We shall use the following result.

Theorem 4.1 (Fischer [9]). *Let* $\mathfrak{a}$ *be a homogeneous ideal of* R *(I.e. if* $p \in \mathfrak{a}$*, then each homogeneous block of* $p \in \mathfrak{a}$*). Let* S *be the space of polynomial solutions of* $a(\partial) f = 0$, $a \in \mathfrak{a}$. *Then* $\mathfrak{a}$, S, R *are vector spaces over* k *and* $R = \mathfrak{a} \otimes S$.

Proof. Let R_m = vector space of homogeneous polynomials of degree m, $0 \leqslant m < \infty$, $\mathfrak{a}_m = R_m \cap \mathfrak{a}$, $S_m = R_m \cap S$. We have $R = \sum_{m=0}^{\infty} \oplus R_m$, with similar expressions for $\mathfrak{a}$ and S. For any two polynomials P, Q, define $(P, Q) = P(\partial)\, Q\,|_{x=0}$. It is readily verified that (P, Q) is an inner product on R with $R_m \perp R_p$ whenever $m \neq p$. We show that $\mathfrak{a}_m$, S_m are orthogonal complements in R_m. Hence $R_m = \mathfrak{a}_m \oplus S_m$, $0 \leqslant m < \infty$, and so $R = \mathfrak{a} \oplus S$. $Q \in S$, $P \in \mathfrak{a}_m \Rightarrow P(\partial)\, Q(x) = 0 \Rightarrow (P, Q) = 0$. Hence $S_m \in \mathfrak{a}_m^{\perp}$. Let $Q \in \mathfrak{a}_m^{\perp}$. We show that $Q \in S_m$. It suffices to check that for any homogeneous $a \in \mathfrak{a}$ of degree $\leqslant m$, $a(\partial)\, Q(x) = 0 \Leftrightarrow b(\partial)\, [a(\partial) Q] = 0$ for all homogeneous b of degree $(m - \deg a)$. Now $b(\partial)\, [a(\partial) Q] = (ba, Q)$. Since $b\, a \in \mathfrak{a}_m$

and $Q \in \mathfrak{a}_m^{\perp}$, we conclude $b(\partial)\left[a(\partial)Q\right] = 0$. Thus $Q \in S_m$, so that $\mathfrak{a}_m^{\perp} \subset S_m$. It follows that $S_m = \mathfrak{a}_m^{\perp}$.

The following lemma will be required for the proof of Theorem 4.2.

LEMMA 4.1. Let $i(x)$ be an invariant of G and $\sigma \in G$. Let $f(x)$ be C^{∞} on an n-dimensional region $\mathscr{R}$. Then $i(\partial) f(\sigma x) = \left[i(\partial) f\right](\sigma x)$, provided $x, \sigma x \in \mathscr{R}$.

Proof. An application of the chain rule yields

$$i(\partial) f(\sigma x) = \left[i(\sigma^{-1}\partial)\right](\sigma x),$$

for any polynomial $i(x)$. If $i(x)$ is invariant under G, then $i(\sigma^{-1}x) = i(x)$, so that $i(\partial) f(\sigma x) = \left[i(\partial) f\right](\sigma x)$.

THEOREM 4.2. (Steinberg [21]). *Let* $\Pi(x) = \prod_{i=1}^{r} L_i(x)$, *where* $L_i(x) = 0$ *are the r.h.'s of* G, *and* $D\,\Pi$ = *linear span of partial derivatives of* $\Pi(x)$. *Let* S *be the solution space of* C^{∞} *functions on the* n*-dimensional region* $\mathscr{R}$ *satisfying* (4.1) $a(\partial) f = 0, x \in \mathscr{R}$ *and* $a \in \mathscr{I}$, $\mathscr{I}$ *being the ideal generated by all homogeneous invariants of* G *of positive degree. Then* $S = D\,\Pi$.

REMARK. If $O(n)$ is the orthogonal group acting on R^n, then it can easily be shown that $x_1^2 + \ldots x_n^2$ is a basis for the invariants of $O(n)$, i.e. each invariant polynomial is a polynomial in $x_1^2 + \ldots + x_n^2$. If we replace G by $O(n)$, then (4.1) reduces to Laplace's equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right) f = 0 .$$

Because of this, it is natural to refer to the elements in S as the harmonic functions for G. Theorem 4.2 describes these harmonic functions.

Proof of Theorem 4.2. The inclusion $D\,\Pi \subset S$ clearly follows from $a(\partial)\,\Pi = 0, a \in \mathscr{I}$. It suffices to prove the latter for a homogeneous invariant of positive degree. By Lemma 3.4, $\Pi(\sigma x) = \det \sigma.\ \Pi(x), \sigma \in G$. By Lemma 4.1, $\left[a(\partial)\Pi\right](\sigma x) = a(\partial)\,\Pi(\sigma x) = \det \sigma \left[a(\partial)\Pi\right]$. Thus $a(\partial)\,\Pi$ is skew. Again by Lemma 3.4, $\Pi \mid a(\partial)\,\Pi$. Since $\deg\left[a(\partial)\Pi\right] < \deg \Pi$, we must have $a(\partial)\,\Pi = 0$.

We now show that $S \subset D\,\Pi$. Let $f \in S$. We prove first that f is a polynomial x_i, $1 \leqslant i \leqslant n$, is a root of $P(X) = \prod_{\sigma \in G} \left[X - x_i(\sigma x)\right] = X^{|G|}$

$+ a_1 X^{|G|-1} + \ldots + a_{|G|}$, where the a_i's are homogeneous invariants of positive degree. Thus $x_i^{|G|} = -a_1 x_i^{|G|-1} \ldots a_{|G|} \in \mathscr{I}$, $1 \leqslant i \leqslant n$. The latter implies that every homogeneous polynomial $a(x)$ of degree $\geqslant n \mid G \mid$ is in $\mathscr{I}$. Hence $a(\partial) f = 0$, whenever $a(x)$ is homogeneous of degree $\geqslant n \mid G \mid \Rightarrow f$ is a polynomial of degree $< n \mid G \mid$. S is therefore a finite dimensional space of polynomials. In view of Fischer's Theorem $S \subset D\Pi \Leftrightarrow (D\Pi)^{\perp} \subset S^{\perp}$. A polynomial $P(x) \in (D\Pi)^{\perp} \Leftrightarrow (P, Q(\partial)\Pi) = 0 \; \forall$ polynomials $Q \Leftrightarrow Q(\partial)\,(P(\partial)\Pi)\,|_{x=0} \; \forall$ polynomials $Q \Leftrightarrow P(\partial)\,\Pi = 0$. We must therefore show that $P(\partial)\,\Pi = 0 \Rightarrow P \in \mathscr{I}$.

It suffices to prove this for homogeneous P. The result holds for deg $P \geqslant n \, |G|$. Suppose that it holds for deg $P = m + 1$. We show that it holds for deg $P = m$ and, by induction, for arbitrary degree. Let $L(x) = 0$ be an r.h. of G. Then $L(\partial)\, P(\partial)\, \Pi(x) = 0$. By the induction hypothesis $L P \in \mathscr{I}$, so that

$$L(x)\, P(x) = \sum_{k=1}^{n} A_k(x)\, I_k(x) \tag{4.2}$$

where the A_k's are polynomials and $I_1, \ldots, I_n$ are a basic set of homogeneous invariants for G. Let σ be the reflection in the r.h. $L(x) = 0$. Substituting σx for x in (4.2) and subtracting the resulting equation from (4.1), we get

$$L(x)\,(P(x) + P(\sigma x)) = \sum_{k=1}^{n} (A_k(x) - A_k(\sigma x))\, I_k(x) \tag{4.3}$$

Each $[A_k(x) - A_k(\sigma x)] = 0$ whenever $L(x) = 0$. Thus

$$L(x) \mid [A_k(x) - A_k(\sigma x)],$$

and

$$P(x) + P(\sigma x) = \sum_{k=1}^{n} \left[\frac{A_k(x) - A_k(\sigma x)}{L(x)}\right] I_k(x) \tag{4.4}$$

shows that $P(x) \equiv -P(\sigma x) \pmod{\mathscr{I}}$. Since the reflections in G generate G, we conclude from the latter that $P(x) \equiv \det \sigma \, P(\sigma x) \pmod{\mathscr{I}}$. Averaging over G, we obtain $P(x) \equiv P^*(x) \pmod{\mathscr{I}}$, where $P^*(x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} \det \sigma \cdot P(\sigma x)$. We claim that $P^*(x)$ is skew. For if $\sigma_1 \in G$, then

$$\begin{aligned} P^*(\sigma_1 x) &= \frac{1}{|G|} \sum_{\sigma \varepsilon G} \det \sigma \cdot P(\sigma \sigma_1 x) \\ &= \frac{1}{\det \sigma_1} \sum_{\sigma \varepsilon G} \det \sigma\sigma_1 \, P(\sigma\sigma_1 x) = \det \sigma_1 \, P^*(x). \end{aligned} \tag{4.5}$$

By lemma 3.4 $P^*(x) = \Pi(x)\, i(x)$, where i is a homogeneous invariant. If $\deg i > 0$, then $P^* \in \mathscr{I} \Rightarrow P \in \mathscr{I}$. Otherwise $P^* = c\,\Pi$, c a constant. By assumption $P(\partial)\,\Pi = 0$, while $a(\partial)\,\Pi = 0$ for $a \in \mathscr{I}$. It follows that $P^*(\partial)\,\Pi = c\,(\Pi, \Pi) \Rightarrow c = 0$, so that $P \equiv 0 \pmod{\mathscr{I}}$.

2. Mean Value Properties

We prove the equivalence of system (4.1) and a certain mean value property.

THEOREM 4.3 (Steinberg [21]). *Let $f(x) \in C$ in the n-dimensional region $\mathscr{R}$ and let it satisfy the mean value property (m.v.p.)*

$$(4.6) \qquad f(x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} f(x + \sigma y), \quad x \in \mathscr{R} \text{ and } \|y\| < \varepsilon_x ,$$

where $\inf\limits_{x \in K} \varepsilon_x > 0$ for any compact subset K of $\mathscr{R}$ and $\|y\|^2 = \sum\limits_{i=1}^{n} y_i^2$. This m.v.p. is equivalent to having $f \in C^\infty$ and satisfying (4.1). It follows from Theorem 4.2 that the space S of continuous solutions to (4.6) $= D\,\Pi$.

REMARK. The harmonic functions on $\mathscr{R}$ are characterized as the continuous functions on $\mathscr{R}$ satisfying the m.v.p. $f(x) = \int f(x+y)\, d\sigma(y)$, $x \in \mathscr{R}$ and $\|y\| < \varepsilon_{x'}$ where $d\sigma(y)$ is the normalized Haar measure on the orthogonal group $O(n)$. (4.6) is just the G-analog of this m.v.p.

Proof of Theorem 4.3. Suppose first that $f(x)$ is C^∞ on $\mathscr{R}$ and satisfies (4.6). Let $a(x)$ be any homogeneous invariant of positive degree. Apply the operator $a(\partial_y)$ to both sides of (4.6). In view of Lemma 4.1, we get

$$(4.7) \qquad 0 = a(\partial_y) f(x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} a(\partial_y) f(x + \sigma y)$$
$$= \frac{1}{|G|} \sum_{\sigma \varepsilon G} \left[a(\partial_y) f(x + y)\right](\sigma y)$$

Use $a(\partial_y) f(x+y) = a(\partial_x) f(x+y)$ and set $y = 0$. We obtain $a(\partial_x) f(x) = 0$, $x \in \mathscr{R}$ and a any homogeneous invariant of positive degree. Hence $a(\partial_x) f(x) = 0$, $x \in \mathscr{R}$ and $a \in \mathscr{I}$. Since $\sum\limits_{i=1}^{n} x_i^2 \in \mathscr{I}$, we conclude in particular that $f(x)$ is harmonic on $\mathscr{R}$.