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## 4. Solomon's Theorem

We present in this section another method for determining the degrees of the basic invariants, valid whenever the underlying field $k$ has characteristic 0.

THEOREM 3.14 (Solomon [18]). *Let $G$ be a finite reflection group acting on the $n$-dimensional space $V$. Let $g_r$ = number of elements of $G$ which fix some $r$-dimensional subspace of $V$ but do not fix a subspace of higher dimension. Let $d_1, \ldots, d_n$ be the degrees of the basic homogeneous invariants of $G$ and set $m_j = d_j - 1$. Then*

$$(3.27) \qquad (t+m_1)\ldots(t+m_n) = g_0 + g_1 t + \ldots + g_n t^n$$

Equating the $t^{n-1}$-coefficients of both sides of (3.27), we obtain $g_1 = r = \sum_{i=1}^{n} m_i$. Setting $t = 1$ in (3.27), we obtain $\prod_{i=1}^{n} (m_i + 1) = \sum_{i=0}^{n} g_i = |G|$. Thus Theorem 3.14 generalizes Theorem 2.2.

To prove Theorem 3.14, we obtain an analog of Molien's formula for the invariant differential forms of $G$. We digress to a brief discussion of differential forms.

For $p > 0$, let $\omega = \sum_{i_1 < \ldots < i_p} r_{i_1 \ldots i_p}(x)\, dx_{i_1} \ldots dx_{i_p}$, where $r_{i_1 \ldots i_p}(x) \in k(x)$, the summation extending over all integer $p$-tuples satisfying $1 \leqslant i_1 < \ldots < i_p \leqslant n$. $\omega$ is called a differential $p$-form (or simply $p$-form). The elements of $k(x)$ are called the 0-forms. If $\eta = \sum_{i_1 < \ldots i_p} s_{i_1 \ldots i_p}(x)\, dx_{i_1} \ldots dx_{i_p}$ is another $p$-form, then we define

$$\omega + \eta = \sum_{i_1 < \ldots < i_p} (r_{i_1 \ldots i_p} + s_{i_1 \ldots i_p})\, dx_{i_1} \ldots dx_{i_p}.$$

Thus the $p$-forms constitute a vector space over $k(x)$ which we denote by $\mathscr{D}_p$. The elements $dx_{i_1} \ldots dx_{i_p}$ form a basis for $\mathscr{D}_p$, so that $\dim \mathscr{D}_p = \binom{n}{p}$, $0 \leqslant p \leqslant n$. We also define a multiplication between two forms as follows. Let $dx_i\, dx_j = -dx_j\, dx_i$; in particular $dx_i\, dx_i = 0$. The product $\omega\eta$ of any two forms $\omega, \eta$ is then obtained by the distributive law. We observe that for 1-forms, $\omega\eta = -\eta\omega$, so that $\omega\omega = 0$. It follows that $\mathscr{D}_p = 0$ for $p > n$. Finally, for any rational function $r$, we define the 1-form $dr$ to be

$$\sum_{i=1}^{n} \frac{\partial r}{\partial x_i} dx_i .$$

It is then readily checked that for $n$ rational functions, $r_1, ..., r_n$, we have

$$dr_1 \dots dr_n = \frac{\partial (r_1, \dots, r_n)}{\partial (x_1, \dots, x_n)} dx_1 \dots dx_n .$$

Let $\sigma$ be a non-singular matrix with entries in $k$. We define

$$\sigma \omega = \sum_{i_1 < \dots < i_p} r_{i_1 \dots i_p} (\sigma^{-1} x)\, dx_{i_1} (\sigma^{-1} x) \dots dx_{i_p} (\sigma^{-1} x)$$

Thus $\sigma$ becomes a linear transformation on each $\mathcal{D}_p$, interpreting the latter as a vector space over $k$. Let $k^n$ be the space of $n$-tuples with entries in $k$. If $G$ is a group of linear transformations acting on $k^n$, then $\omega$ is said to be invariant under $G$ provided $\sigma\omega = \omega$, $\forall \sigma \in G$.

We shall prove Theorem 3.14 describing the invariant differential forms with polynomial coefficients. $G$ is assumed throughout to be a finite reflection group acting on $k^n$.

LEMMA 3.4. Let $I_1, ..., I_n$ be basic homogeneous invariants for $G$. Let

$$\Pi (x) = \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)} .$$

The polynomial $p(x)$ satisfies $\sigma p = (\det \sigma) p$, for every $\sigma \in G$ (in which case, we say $p$ is skew) iff $p = \Pi i$ where $i$ is a polynomial invariant under $G$.

*Proof.* Let $y = \sigma x$. Then

$$\Pi (x) = \frac{\partial (I_1 (y), \dots, I_n(y))}{\partial (x_1, \dots, x_n)} \tag{3.28}$$

$$= \frac{\partial (I_1 (y), \dots, I_n (y))}{\partial (y_1, \dots, y_n)} \det \sigma = \Pi (\sigma x) \det \sigma$$

which shows that $\Pi$ is skew. Hence $\Pi i$ is skew for every invariant polynomial $i$.

Conversely, let $p(x)$ be skew. Let $\pi$ be an r.h. of $G$ with equation $L(x) = 0$. By Lemma 2.2, we may choose $v \notin \pi$, so that $v$ is a common eigenvector to all reflections in $G$ with r.h. $\pi$. Choose $x = Ty$, $\det T \neq 0$, so that in the $y$ coordinates the equation of $\pi$ becomes $y_n = 0$ and $v$ becomes $(0, ..., 0, 1)$. Let $q(y) = p(Ty)$. Let $H$ be the subgroup of $G$ which fixes $\pi$. By Lemma 2.2, $H$ is a cyclic group. Let $\sigma$ generate $H$ and $h = \text{ord}\, H$. If $\zeta$ is the eigenvalue of $\sigma$ which is a primitive $h$-th root of 1, then

$q(y_1, \ldots, y_{n-1}, \zeta y_n) = \zeta^{-1} q(y_1, \ldots, y_n)$. Writing $q = \sum q_i y_n^i$, the $q_i$'s being polynomials in $y_1, \ldots, y_{n-1}$, we obtain

$$\Sigma q_i \zeta^{i+1} y_n^i = \Sigma q_i y_n^i \tag{3.29}$$

Equating coefficients in (3.29), we conclude $q_i = 0$ whenever $h \nmid i+1$. Thus $q_i = 0$ for $i < h-1 \Rightarrow y_n^{h-1} \mid q \Rightarrow L^{h-1} \mid p$. Repeating this argument for all r.h.'s of $G$ and using Theorem 2.5, we conclude that $P = \Pi i$, where $i$ is a polynomial. $\sigma i = \sigma P / \sigma \Pi = \dfrac{P}{\Pi} = i$ shows that $i$ is invariant under $G$.

LEMMA 3.5. Let $\sigma$ be a non-singular matrix with entries in $k$. Let $r \in k(x)$. Then $\sigma(dr) = d(\sigma r)$.

*Proof.* By definition

$$\sigma(dr) = \sum_{i=1}^{n} \frac{\partial r}{\partial x_i}(\sigma^{-1}x)\, dx_i(\sigma^{-1}x), \quad d(\sigma r) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(r(\sigma^{-1}x))\, dx_i \tag{3.30}$$

Let $\sigma^{-1} = (a_{ij})$. Then $x_i(\sigma^{-1}x) = \sum_{j=1}^{n} a_{ij} x_j$ and $\dfrac{\partial x_i}{\partial x_j}(\sigma^{-1}x) = a_{ij}$.

Hence

$$dx_i(\sigma^{-1}x) = \sum_{j=1}^{n} a_{ij}\, dx_j \tag{3.31}$$

Applying the chain rule,

$$\frac{\partial}{\partial x_i}(r(\sigma^{-1}x)) = \sum_{j=1}^{n} \frac{\partial r}{\partial x_j}(\sigma^{-1}x)\, a_{ji} \tag{3.32}$$

Inserting (3.31), (3.32) into (3.30), we get $\sigma(dr) = d(\sigma r)$.

THEOREM 3.15. *Every invariant p-form with polynomial coefficients may be expressed uniquely as*

$$\sum_{i_1 < \ldots < i_p} a_{i_1 \ldots i_p}\, dI_{i_1} \ldots dI_{i_p},\; a_{i_1 \ldots i_p} \in k[I_1, \ldots, I_n].$$

*Proof.* By Lemma 3.5, $\sigma(dI_k) = dI_k$, so that $dI_1, \ldots, dI_n$ are invariant forms. Since $\sigma(\omega\eta) = \sigma(\omega)\,\sigma(\eta)$ for any two forms $\omega, \eta$, we conclude that $\sum_{i_1 < \ldots < i_p} a_{i_1 \ldots i_p}\, dI_{i_1} \ldots dI_{i_p}$ is invariant whenever $a_{i_1 \ldots 1_p} \in k(I_1, \ldots, I_n)$.

We show that the $\binom{n}{p}$ forms $dI_{i_1} \ldots dI_{i_p}$ are linearly independent over $k(x)$, so that they form a basis for $\mathcal{D}_p$ over $k(x)$. Suppose that

$$\sum_{i_1 < \ldots < i_p} k_{i_1 \ldots i_p}\, dI_{i_1} \ldots dI_{i_p} = 0,\; k_{i_1 \ldots i_p} \in k(x).$$

Multiply this relation by $dI_{i_{p+1}} \ldots dI_{i_n}$, where $i_{p+1}, \ldots, i_n$ are the indices complementary to $i_1, \ldots, i_p$. We obtain

$$k_{i_1 \ldots i_p} dI_1 \ldots dI_n = k_{i_1 \ldots i_p} \Pi(x)\, dx_1 \ldots dx_n = 0 \Rightarrow k_{i_1 \ldots i_p} = 0$$

for all $i_1, \ldots, i_p$. Hence the $\binom{n}{p}$ forms $dI_{i_1} \ldots dI_{i_p}$ are linearly independent over $k(x)$. It follows that every $p$-form $\omega$ may be expressed uniquely as

$$\omega = \sum_{i_1 < \cdots < i_p} a_{i_1 \ldots i_p} dI_{i_1} \ldots dI_{i_p}, \; a_{i_1 \ldots i_p} \in k(x).$$

If $\omega$ is invariant, then the group averaging argument shows that $a_{i_1 \ldots i_p} \in k(I_1, \ldots, I_n)$. Multiply both sides of the above relation by $dI_{i_{p+1}} \ldots dI_{i_n}$. We get

$$\omega\, dI_{i_{p+1}} \ldots dI_{i_n} = \pm \Pi\, a_{i_1 \ldots i_p} dx_1 \ldots dx_n . \tag{3.33}$$

Let $\omega$ be a $p$-form with polynomial coefficients. We conclude from (3.33) that $\Pi\, a_{i_1 \ldots i_p}$ is a polynomial. Since $\Pi\, a_{i_1 \ldots i_p}$ is skew, Lemma 3.4 implies that $\Pi\, a_{i_1 \ldots i_p} = \Pi\, i$, $i$ being an invariant polynomial. Hence $a_{i_1 \ldots i_p} \in k[I_1, \ldots, I_n]$ for all $i_1, \ldots, i_p$, thus proving Theorem 3.11.

THEOREM 3.16. *Let $\sigma_p(x_1, \ldots, x_n)$ be the $p$-th elementary symmetric function in $x_1, \ldots, x_n$ ($\sigma_0$ is interpreted to be 1). Let $\omega_1(\gamma), \ldots, \omega_n(\gamma)$ be the eigenvalues of $\gamma$, $\gamma \in G$. Then*

$$\frac{\sigma_p(t^{m_1}, \ldots, t^{m_n})}{(1-t^{m_1+1}) \ldots (1-t^{m_n+1})} = \frac{1}{|G|} \sum_{\gamma \in G} \frac{\sigma_p(\omega_1(\gamma), \ldots, \omega_n(\gamma))}{(1-\omega_1(\gamma)t) \ldots (1-\omega_n(\gamma)t)}, \quad 0 \leqslant p \leqslant n \tag{3.34}$$

REMARK. For $p = 0$, the above becomes formula (2.5) of Chapter II.

*Proof.* Let $\mathscr{D}_{pm}$ = space of $p$-forms whose coefficients are homogeneous polynomials of degree $m$. $\mathscr{D}_{pm}$ is a finite dimensional vector space over $k$. Let $\mathscr{I}_{pm}$ = space of invariant forms in $\mathscr{D}_{pm}$ and $d_{pm} = \dim \mathscr{I}_{pm}$. For $0 \leqslant p \leqslant n$, let $\mathfrak{p}_p(t) = \sum_{m=0}^{\infty} d_{pm} t^m$. We obtain two formulas for $\mathfrak{p}_p(t)$ by computing $d_{pm}$ in two different ways. By Theorem 3.15, the differentials

$$I_1^{k_1} \ldots I_n^{k_n} dI_{i_1} \ldots dI_{i_p}, \; m = k_1(m_1+1) \ldots + k_n(m_n+1) + m_{i_1} + \ldots + m_{i_p},$$

form a basis for $\mathcal{J}_{pm}$, so that

$$(3.35) \qquad \mathfrak{p}_p(t) = \frac{\sigma_p(t^{m_1}, \ldots, t^{m_n})}{(1-t^{m_1+1}) \ldots (1-t^{m_n+1})}$$

Let $\tilde{k}$ = algebraic closure of $k$. Define $\tilde{\mathcal{D}}_{pm}$, $\tilde{\mathcal{J}}_{pm}$, analogously to $\mathcal{D}_{pm}$, $\mathcal{J}_{pm}$, replacing $k$ by $\tilde{k}$. For $\gamma \in G$, $\gamma$ acts both on $\mathcal{D}_{pm}$ and $\tilde{\mathcal{D}}_{pm}$. Let $(\mathrm{Tr}\ \gamma)_{pm}$ = trace of $\gamma$ as a transformation on $\mathcal{D}_{pm}$ = trace of $\gamma$ as a transformation on $\tilde{\mathcal{D}}_{pm}$. By Lemma 1.2

$$(3.36) \qquad d_{pm} = \frac{1}{|G|} \sum_{\gamma \varepsilon G} (Tr\,\gamma)_{pm}$$

Choose $T$ so that $T \sigma T^{-1} = D$, $D$ being diagonal with diagonal entries $\omega_1(\gamma), \ldots, \omega_n(\gamma)$. The elements $x^a\, d x_{i_1} \ldots d x_{i_p}$, $|a| = m$ and $1 \leqslant i_1 < \ldots < i_p \leqslant n$, form a basis for $\tilde{\mathcal{D}}_{pm}$. Since

$$(3.37) \qquad D(x^a dx_i \ldots dx_{i_p}) = [\omega(\gamma^{-1})]^a \omega_{i_1}(\gamma^{-1}) \ldots \omega_{i_p}(\gamma^{-1}),$$

we have

$$(3.38) \qquad (TrD)_{pm} = \sum_{|a|=m} [\omega(\gamma^{-1})]^m \sigma_p(\omega(\gamma^{-1}))$$

(3.36), (3.38) yield

$$(3.39) \qquad d_{pm} = \frac{1}{|G|} \sum_{\gamma \varepsilon G} \sum_{|a|=m} [\omega(\gamma)]^a \sigma_p[\omega(\gamma)]$$

so that

$$(3.40) \qquad \mathfrak{p}_p(t) = \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{r \varepsilon G} \sum_{|a|=m} [\omega(\gamma)]^a \sigma_p(\omega(\gamma))\, t^m$$

$$= \frac{1}{|G|} \sum_{\gamma \varepsilon G} \frac{\sigma_p(\omega(\gamma))}{(1-\omega_1(\gamma)t) \ldots (1-\omega_n(\gamma)t)}$$

(3.34) follows from (3.35) and (3.40).

We derive from (3.34) the following identity.

THEOREM 3.17. *For* $1 \leqslant p \leqslant n$,

$$(3.41) \qquad \sum_{i_1 < \ldots < i_p} \frac{t^{m_{i_1} + \ldots + m_{i_p}}}{(1-t^{m_{i_1}+1}) \ldots (1-t^{m_{i_p}+1})}$$

$$= \frac{1}{|G|} \sum_{\gamma \varepsilon G} \sum_{i_1 < \ldots < i_p} \frac{\omega_{i_1}(\gamma) \ldots \omega_{i_p}(\gamma)}{(1-\omega_{i_1}(\gamma)t) \ldots (1-\omega_{i_p}(\gamma)t)}$$

*Proof.* One verifies readily, for $1 \leqslant p \leqslant n$, the identity

$$(3.42) \qquad \sum_{i_1 < \ldots < i_p} \frac{u_{i_1} \ldots u_{i_p}}{(1-u_{i_1}t) \ldots (1-u_{i_p}t)}$$

$$= \frac{h_{p1}(t)\,\sigma_1(u_1, \ldots, u_n) + \ldots + h_{pn}(t)\,\sigma_n(u_1, \ldots, u_n)}{(1-u_it) \ldots (1-u_nt)}$$

the $u_i$'s being indeterminates and the $h_{pi}$'s being polynomials in $t$. Substitute for $u_i$, $\omega_i(\gamma)$ and average over the group. By Theorem 3.16, the group average becomes expression (3.42), $u_i$ being replaced by $t^{m_i}$, thus proving (3.41).

We can now provide the

*Proof of Theorem 3.14.* Expand both sides of (3.41) in powers of $1-t$ and equate the coefficients of $(1-t)^{-p}$. For the left side this coefficient is

$$\sum_{i_1 < \ldots < i_p} \frac{1}{(m_{i_1}+1) \ldots (m_{i_p}+1)}$$

Let $\gamma$ be an element which fixes an $r$ dimensional subspace, but does not fix a higher dimensional subspace. This means that precisely $r$ of the eigenvalues of $\gamma$ equal 1. $\gamma$ contributes to the coefficient of $(1-t)^{-p}$ on the right side of (3.41) iff $r \geqslant p$, the contribution being $\binom{r}{p}$. It follows that for the right side, the $(1-t)^{-p}$ coefficient is $\frac{1}{|G|} \sum_{r=0}^{n} \binom{r}{p} g_r$. Since $\prod_{i=1}^{n} (m_i+1) = |G|$, we conclude that

$$(3.43) \qquad \sum_{r=0}^{n} \binom{r}{p} g_r = \sum_{i_1 < \ldots < i_{n-p}} (m_{i_1}+1) \ldots (m_{i_{n-p}}+1), \quad 1 \leqslant p \leqslant n$$

Note that for $p = 0$, (3.43) becomes $|G| = (m_1+1) \ldots (m_n+1)$. Hence (3.43) also holds for $p = 0$.

The left and right side of (3.43) equal respectively $\frac{1}{p!}$ ($p$-th derivative at $t = 1$) of $g_0 + \ldots + g_n t^n$, $(t+m_1) \ldots (t+m_n)$. Thus $(t+m_1) \ldots (t+m_n) = g_0 + \ldots + g_n t^n$.