Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 24 (1978)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INVARIANTS OF FINITE REFLECTION GROUPS
Autor: Flatto, Leopold
Kapitel: 3. Tabulation of the Degrees
DOI: https://doi.org/10.5169/seals-49704

3. Tabulation of the Degrees

Theorem 3.8 can be used to compute the degrees of the basic homogeneous invariants of G, in case G is an irreducible reflection group acting on R^n. This has been done in [7], and we tabulate these degrees below

Group	$d_1, \ldots, d_n$
A_n $(n \geqslant 1)$	$2, \ldots, n+1$
B_n $(n \geqslant 2)$	$2, 4, \ldots, 2n$
D_n $(n \geqslant 4)$	$2, 4, \ldots, n, \ldots, 2n-4, 2n-2$
H_2^n $(n \geqslant 5)$	$2, n$
E_6	2, 5, 6, 8, 9, 12
E_7	2, 6, 8, 10, 12, 14, 18
E_8	2, 8, 12, 14, 18, 20, 24, 30
F_4	2, 6, 8, 12
I_3	2, 6, 10
I_4	2, 12, 20, 30

We observe that in each case, $d_1 = 2$. This can be seen as follows. Suppose that there existed a homogeneous invariant $I(x)$ of degree 1. Since $I(\sigma x) = I(x)$ whenever $\sigma \in G$, the hyperplane $\{x \mid I(x) = 0\}$ would be a proper invariant subspace of G, contradicting that the latter is irreducible. Hence there are no homogeneous invariants of degree 1 and $d_1 \geqslant 2$. On the other hand, $\sum_{i=1}^{n} x_i^2$ is invariant under G as G is orthogonal. It follows that $d_1 = 2$, with corresponding invariant $I_1 = \sum_{i=1}^{n} x_i^2$.

In applying Theorem 3.8, we must find the roots of the characteristic equation (3.23). In some cases, this is a rather tedious computation. For the groups A_n, B_n, D_n H_2^n we can exhibit a basis of homogeneous invariants without the use of Theorem 3.8. We require

Theorem 3.13. *Let G be a finite reflection group acting on the n-dimensional vector space V over a given field k. Let $P_1, \ldots, P_n$ be homogeneous* *invariants of* G *of respective degrees* $k_1, ..., k_n$. $P_1, ..., P_n$ *form a basis for the invariants of* $G \Leftrightarrow k_1 ... k_{\cdot n} = |G|$ *and*

$$\Delta = \frac{\partial (P_1, ..., P_n)}{\partial (x_1, ..., x_n)} \neq 0 .$$

Proof. By relabeling indices, we may assume $k_1 \leqslant ... \leqslant k_n$. The $\Rightarrow$ part of the theorem is contained in Theorems 1.2, 2.2, 2.3. Conversely, let $k_1 ... k_n = |G|$ and $\Delta \neq 0$. Thus $P_1, ..., P_n$ are algebraically independent. Let $I_1, ..., I_n$ be basic homogeneous invariants of respective degrees $d_1, ..., d_n$. Suppose $k_i = d_i$, $1 \leqslant i \leqslant i_0$, but $k_{i_0+1} < d_{i_0+1}$. Then $P_1, ..., P_{i_0+1}$ are polynomials in $I_1, ..., I_{i_0}$, implying that $P_1, ..., P_n$ are algebraically dependent, a contradiction. Hence $k_i \geqslant d_i$, $1 \leqslant i \leqslant n$. Since $\prod_{i=1}^{n} d_i = \prod_{i=1}^{n} k_i = |G|$, we must have $k_i = d_i$, $1 \leqslant i \leqslant n$.

Let $\delta_m = \dim \mathscr{I}_m$, $0 \leqslant m < \infty$, $\mathscr{I}_m$ being the space of homogeneous invariants of degree m. Then δ_m = number of non-negative integral solutions to $j_1 d_1 + ... + j_n d_n = m$. This number also equals the number of monomials $P_1^{j1} ... P_1^{jn}$ which are of degree m. The algebraic independence of $P_1, ..., P_n$ implies that these δ_m monomials are linearly independent over k. Thus $\mathscr{I}_m$ is spanned by these monomials for $0 \leqslant m < \infty$. We have shown that every homogeneous invariant is a polynomial in $P_1, ..., P_n$, so that the P_i's form a basis for the invariants of G.

We now obtain an explicit basis for the invariants of A_n, B_n, D_n, H_2^n. A_n: This group consists of the $(n+1)!$ permutations $x_i' = x_{\sigma(i)}$, $1 \leqslant i \leqslant n+1$, restricted to the subspace $V = \{x \mid x_1 + ... + x_{n+1} = 0\}$. We choose $x_1, ..., x_n$ as coordinates on V. Let $P_i = \sum_{j=1}^{n+1} x_j^{i+1}$, $1 \leqslant i \leqslant n$, where $x_{n+1} = -(x_1 + ... + x_n)$. P_i is a homogeneous invariant of degree $i+1$. We have $2 \cdot ... \cdot (n+1) = (n+1)! = |A_n|$.

We show that $\Delta \neq 0$. Now

$$\frac{\partial P_i}{\partial x_j} = (i+1)\, x_j^i - (i+1)\, x_{n+1}^i, \; 1 \leqslant i,j \leqslant n .$$

Hence $\Delta = (n+1)!\, D$ where D is the $n \times n$ determinant whose (ij)-th entry $= x_j^i - x_{n+1}^i$. To evaluate D, we introduce the Vandermonde determinant

$$\begin{vmatrix} 1 & \cdots\cdots & 1 \\ x_1 & \cdots\cdots & x_{n+1} \\ x_1^n & \cdots\cdots & x_{n+1}^n \end{vmatrix} = \prod_{1 \leqq i < j \leqq n+1} (x_j - x_i)$$

Subtracting the $(n+1)$-th column from the first n columns, the above determinant is readily seen to equal $(-1)^n D$. Thus

$$(3.25) \qquad \Delta = (-1)^{n+2}(n+1)! \prod_{1 \leq i < j \leq n+1} (x_j - x_i) =$$

$$(n+1)! \prod_{1 \leq j \leq n} (x_j - x_i) \cdot \prod_{i=1}^{n} (x_i + s)$$

where $s = x_1 + \ldots + x_n$. (3.25) shows that $\Delta \neq 0$. We conclude that $d_1 = 2, \ldots, d_n = n+1$.

B_n: Let $P_i = \sum_{j=1}^{n} x_j^{2i}$, $1 \leqslant i \leqslant n$. P_i is a homogeneous invariant of degree $2i$. We have $2 \cdot \ldots \cdot 2n = 2^n n! = |B_n|$. A computation shows that $\Delta = 2^n n! \prod_{i=1}^{n} x_i \prod_{1 \leq i < j \leq n} (x_j^2 - x_i^2) \neq 0$. It follows that $d_1 = 2, \ldots, d_n = 2n$.

D_n: Let $P_1 = x_1 \ldots x_n$, $P_i = \sum_{j=1}^{n} x_j^{2(i-1)}$, $2 \leqslant i \leqslant n$. P_1 is a homogeneous invariant of degree n; P_i, $2 \leqslant i \leqslant n$, is a homogeneous invariant of degree $2(i-1)$. The product of the degrees $= n \cdot 2 \cdot 4 \cdot \ldots \cdot (2n-2) = 2^{n-1} n! = |D_n|$.

$$(3.26) \qquad \Delta = \begin{vmatrix} \dfrac{P_1}{x_1} & \cdots & \dfrac{P_1}{x_n} \\ 2x_1 & \cdots & 2x_n \\ \cdot & \cdots & \cdot \\ 2(n-1)x_1^{2n-3} & \cdots & 2(n-1)x_n^{2n-3} \end{vmatrix}$$

$$= 2^{n-1}(n-1)! \prod_{1 \leq i < j \leq n} (x_j^2 - x_i^2) \neq 0$$

It follows that $d_i, \ldots, d_n$ are identical with the numbers $2, 4, \ldots, n, \ldots, 2n-4, 2n-2$.

H_2^n: Let z be the complex coordinate $x_1 + i x_2$. H_2^n may be described as the group generated by the transformation $z \to \bar{z}$, $z \to \zeta z$, where $\zeta = e^{\frac{2\pi i}{n}}$. Let $P_1 = x_1^2 + x_2^2$, $P_2 = Re\, z^n$. P_1, P_2 are homogeneous invariants of respective degrees $2, n$. The product of these degrees $= 2n = |H_2^n|$. A computation yields

$$\frac{\partial(P_1, P_2)}{\partial(x_1, x_2)} = -2n\, Im\, z^n \neq 0 .$$

It follows that $d_1 = 2$, $d_2 = n$.