

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 24 (1978)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INVARIANTS OF FINITE REFLECTION GROUPS
Autor: Flatto, Leopold
Kapitel: 3. Tabulation of the Degrees
DOI: <https://doi.org/10.5169/seals-49704>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

3. TABULATION OF THE DEGREES

Theorem 3.8 can be used to compute the degrees of the basic homogeneous invariants of G , in case G is an irreducible reflection group acting on R^n . This has been done in [7], and we tabulate these degrees below

Group	d_1, \dots, d_n
A_n ($n \geq 1$)	$2, \dots, n + 1$
B_n ($n \geq 2$)	$2, 4, \dots, 2n$
D_n ($n \geq 4$)	$2, 4, \dots, n, \dots, 2n - 4, 2n - 2$
H_2^n ($n \geq 5$)	$2, n$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$
F_4	$2, 6, 8, 12$
I_3	$2, 6, 10$
I_4	$2, 12, 20, 30$

We observe that in each case, $d_1 = 2$. This can be seen as follows. Suppose that there existed a homogeneous invariant $I(x)$ of degree 1. Since $I(\sigma x) = I(x)$ whenever $\sigma \in G$, the hyperplane $\{x \mid I(x) = 0\}$ would be a proper invariant subspace of G , contradicting that the latter is irreducible. Hence there are no homogeneous invariants of degree 1 and $d_1 \geq 2$. On the other hand, $\sum_{i=1}^n x_i^2$ is invariant under G as G is orthogonal. It follows

that $d_1 = 2$, with corresponding invariant $I_1 = \sum_{i=1}^n x_i^2$.

In applying Theorem 3.8, we must find the roots of the characteristic equation (3.23). In some cases, this is a rather tedious computation. For the groups A_n, B_n, D_n, H_2^n we can exhibit a basis of homogeneous invariants without the use of Theorem 3.8. We require

THEOREM 3.13. *Let G be a finite reflection group acting on the n -dimensional vector space V over a given field k . Let P_1, \dots, P_n be homogeneous*

invariants of G of respective degrees k_1, \dots, k_n . P_1, \dots, P_n form a basis for the invariants of $G \Leftrightarrow k_1 \dots k_n = |G|$ and

$$\Delta = \frac{\partial(P_1, \dots, P_n)}{\partial(x_1, \dots, x_n)} \neq 0.$$

Proof. By relabeling indices, we may assume $k_1 \leq \dots \leq k_n$. The \Rightarrow part of the theorem is contained in Theorems 1.2, 2.2, 2.3. Conversely, let $k_1 \dots k_n = |G|$ and $\Delta \neq 0$. Thus P_1, \dots, P_n are algebraically independent. Let I_1, \dots, I_n be basic homogeneous invariants of respective degrees d_1, \dots, d_n . Suppose $k_i = d_i, 1 \leq i \leq i_0$, but $k_{i_0+1} < d_{i_0+1}$. Then P_1, \dots, P_{i_0+1} are polynomials in I_1, \dots, I_{i_0} , implying that P_1, \dots, P_n are algebraically dependent, a contradiction. Hence $k_i \geq d_i, 1 \leq i \leq n$. Since

$$\prod_{i=1}^n d_i = \prod_{i=1}^n k_i = |G|, \text{ we must have } k_i = d_i, 1 \leq i \leq n.$$

Let $\delta_m = \dim \mathcal{J}_m$, $0 \leq m < \infty$, \mathcal{J}_m being the space of homogeneous invariants of degree m . Then $\delta_m = \text{number of non-negative integral solutions to } j_1 d_1 + \dots + j_n d_n = m$. This number also equals the number of monomials $P_1^{j_1} \dots P_n^{j_n}$ which are of degree m . The algebraic independence of P_1, \dots, P_n implies that these δ_m monomials are linearly independent over k . Thus \mathcal{J}_m is spanned by these monomials for $0 \leq m < \infty$. We have shown that every homogeneous invariant is a polynomial in P_1, \dots, P_n , so that the P_i 's form a basis for the invariants of G .

We now obtain an explicit basis for the invariants of A_n, B_n, D_n, H_2^n . A_n : This group consists of the $(n+1)!$ permutations $x'_i = x_{\sigma(i)}$, $1 \leq i \leq n+1$, restricted to the subspace $V = \{x \mid x_1 + \dots + x_{n+1} = 0\}$.

We choose x_1, \dots, x_n as coordinates on V . Let $P_i = \sum_{j=1}^{n+1} x_j^{i+1}$, $1 \leq i \leq n$, where $x_{n+1} = -(x_1 + \dots + x_n)$. P_i is a homogeneous invariant of degree $i+1$. We have $2 \cdot \dots \cdot (n+1) = (n+1)! = |A_n|$.

We show that $\Delta \neq 0$. Now

$$\frac{\partial P_i}{\partial x_j} = (i+1)x_j^i - (i+1)x_{n+1}^i, \quad 1 \leq i, j \leq n.$$

Hence $\Delta = (n+1)! D$ where D is the $n \times n$ determinant whose (ij) -th entry $= x_j^i - x_{n+1}^i$. To evaluate D , we introduce the Vandermonde determinant

$$\begin{vmatrix} 1 & \dots & \dots & 1 \\ x_1 & \dots & \dots & x_{n+1} \\ x_1^n & \dots & \dots & x_{n+1}^n \end{vmatrix} = \prod_{1 \leq i < j \leq n+1} (x_j - x_i)$$

Subtracting the $(n+1)$ -th column from the first n columns, the above determinant is readily seen to equal $(-1)^n D$. Thus

$$(3.25) \quad \Delta = (-1)^{n+2} (n+1)! \prod_{1 \leq i < j \leq n+1} (x_j - x_i) = \\ (n+1)! \prod_{1 \leq j \leq n} (x_j - x_i) \cdot \prod_{i=1}^n (x_i + s)$$

where $s = x_1 + \dots + x_n$. (3.25) shows that $\Delta \neq 0$. We conclude that $d_1 = 2, \dots, d_n = n+1$.

B_n : Let $P_i = \sum_{j=1}^n x_j^{2i}$, $1 \leq i \leq n$. P_i is a homogeneous invariant of degree $2i$. We have $2 \cdot \dots \cdot 2n = 2^n n! = |B_n|$. A computation shows that $\Delta = 2^n n! \prod_{i=1}^n x_i \prod_{1 \leq i < j \leq n} (x_j^2 - x_i^2) \neq 0$. It follows that $d_1 = 2, \dots, d_n = 2n$.

D_n : Let $P_1 = x_1 \dots x_n$, $P_i = \sum_{j=1}^n x_j^{2(i-1)}$, $2 \leq i \leq n$. P_1 is a homogeneous invariant of degree n ; P_i , $2 \leq i \leq n$, is a homogeneous invariant of degree $2(i-1)$. The product of the degrees $= n \cdot 2 \cdot 4 \cdot \dots \cdot (2n-2) = 2^{n-1} n! = |D_n|$.

$$(3.26) \quad \Delta = \begin{vmatrix} \frac{P_1}{x_1} & \dots & \frac{P_1}{x_n} \\ 2x_1 & \dots & 2x_n \\ \cdot & \dots & \cdot \\ 2(n-1)x_1^{2n-3} & \dots & 2(n-1)x_n^{2n-3} \end{vmatrix} \\ = 2^{n-1}(n-1)! \prod_{1 \leq i < j \leq n} (x_j^2 - x_i^2) \neq 0$$

It follows that d_1, \dots, d_n are identical with the numbers $2, 4, \dots, n, \dots, 2n-4, 2n-2$.

H_2^n : Let z be the complex coordinate $x_1 + i x_2$. H_2^n may be described as the group generated by the transformation $z \rightarrow \bar{z}$, $z \rightarrow \zeta z$, where $\zeta = e^{\frac{2\pi i}{n}}$. Let $P_1 = x_1^2 + x_2^2$, $P_2 = \operatorname{Re} z^n$. P_1, P_2 are homogeneous invariants of respective degrees 2, n . The product of these degrees $= 2n = |H_2^n|$. A computation yields

$$\frac{\partial (P_1, P_2)}{\partial (x_1, x_2)} = -2n \operatorname{Im} z^n \neq 0.$$

It follows that $d_1 = 2, d_2 = n$.