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Autor: Flatto, Leopold
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- 3) Each G_i , $1 \leq i \leq k$, is one of the groups described in Theorem 3.5. G is a Coxeter group iff $V_0 = 0$.

The proof of Theorem 3.6 is identical with that of Theorem 2.7. We simply observe that we may now choose the V_i 's to be mutually orthogonal.

2. THE COMPUTATION OF THE DEGREES FOR REAL FINITE REFLECTION GROUPS

Let G be a finite irreducible orthogonal reflection group acting on the n -dimensional Euclidean space R^n . Let F be a fundamental region as described in Theorem 3.3 and R_1, \dots, R_n the n reflections in the walls of F . We shall relate the degrees d_1, \dots, d_n of the basic homogeneous invariants to the eigenvalues of $R_1 \dots R_n$. We first prove

THEOREM 3.7. *Let $\sigma(i)$ be any permutation of $1, \dots, n$. Then $R_1 \dots R_n$ is conjugate to $R_{\sigma(1)} \dots R_{\sigma(n)}$*

Proof. Observe that $R_1 (R_1 \dots R_n) R_1 = R_2 \dots R_n R_1$ so that all cyclic permutations yield conjugate transformations. We may also permute any two adjacent R_i 's for which the corresponding walls are orthogonal, as the R_i 's then commute. Theorem 3.7 will then follow from the following

LEMMA 3.1. Let p_1, \dots, p_n be nodes of a tree T . Any circular arrangement of $1, \dots, n$ can be obtained from a sequence of interchanges of pairs i, j which are adjacent on the circle and for which p_i, p_j are not linked in T .

Proof of Lemma 3.1. We proceed by induction, the result being obvious for $n = 1$ or 2 . We may assume that p_n is an end node of the tree, i.e. it links to precisely one other node. We first rearrange $1, \dots, n - 1$ as we wish. To show that this can be done, we just consider the possibility $---inj---$ where p_i, p_j are not linked. If p_i, p_n are not linked, then we interchange first i, n and then i, j , obtaining $---nji---$. If p_j, p_n are not linked, then we first interchange j, n and then j, i , obtaining $---jin---$. We may therefore arrange $1, \dots, n - 1$ in the desired order. Shifting n in one direction, which is permissible as n just fails to commute with one element, we obtain the desired arrangement of $1, \dots, n$.

In view of Theorem 3.7, the eigenvalues of $R_1 \dots R_n$ are independent of the order in which the R_i 's appear. They are also independent of the particularly chosen F . For let F' be another fundamental region as described in Theorem 3.3. Then $F' = \sigma F$, $\sigma \in G$. The reflections in the walls of F'

are given by $R'_i = \sigma R_i \sigma^{-1}$, $1 \leq i \leq n$, so that $R'_1 \dots R'_n = \sigma R_1 \dots R_n \sigma^{-1}$. The main result of the present section is the following

THEOREM 3.8 (Coleman [8]). *Let $R_1 \dots R_n$ have order h . Let $\zeta = e^{2\pi i/h}$. The eigenvalues of $R_1 \dots R_n$ are given by $\zeta^{(d_j-1)}$, $1 \leq j \leq n$, the d_j 's being the degrees of the basic homogeneous invariants of G .*

Theorem 3.8. was first obtained by Coxeter [7], who verified this fact for each group listed in Theorem 3.5. Coleman [8] supplied a general proof, using the fact that the number of reflections = $\frac{1}{2} nh$. This fact, which was at first known only by individual verification [7], was proven by Steinberg [20]. In view of Theorem 3.8, the numbers $m_j = d_j - 1$ are usually referred to as the exponents of the group G .

We begin by proving Steinberg's result, needed for the proof of Coleman's theorem. We require a preliminary lemma and employ the following terminology. Let $A = (a_{ij})$ be an $n \times n$ matrix with non-negative entries. We associate with A a graph \mathcal{G} consisting of n nodes, connecting the nodes i, j iff $a_{ij} > 0$. A is said to be connected iff \mathcal{G} is connected.

LEMMA 3.2. Let $A = (a_{ij})$ be a symmetric connected matrix. The largest eigenvalue λ of A is positive and a corresponding eigenvector e can be chosen all of whose entries are positive.

REMARK. The above is a special case of a theorem of Frobenius concerning the eigenvalues of matrices with non-negative entries [13]. Indeed the symmetry of A is not required. This extraneous assumption permits for a somewhat simpler proof and suffices for our purposes.

Proof. Let $Q(x) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$ be the quadratic form associated with (a_{ij}) . Then $\lambda = \underset{\|x\|=1}{\text{Max}} Q(x) > 0$, where $\|x\|^2 = \sum_{i=1}^n x_i^2$. Choose $v = (v_1, \dots, v_n)$, $\|v\| = 1$, so that $Q(v) = \lambda$ and let $e = (e_1, \dots, e_n)$, where $e_i = |v_i|$, $1 \leq i \leq n$. Then $e_i \geq 0$, $1 \leq i \leq n$, and $\|e\| = 1$. As all $a_{ij} \geq 0$ and $\|e\| = 1$, we have $\lambda = Q(v) \leq Q(e) \leq \lambda$, so that $Q(e) = \lambda$. The latter implies $Ae = \lambda e$. It remains to show that each $e_i > 0$. Choose $e_j > 0$. Because of the connectivity assumption, we may choose $i_1, \dots, i_r = j$ so that $a_{i_1 j_1}, a_{j_1 j_2}, \dots, a_{j_{r-1} j}$ are all > 0 . The relation $\lambda e_{j_{r-1}} = \sum_{k=1}^n a_{j_{r-1} k} e_k$ shows that $e_{j_{r-1}} > 0$. Repeating this reasoning r times, we conclude that each $e_i > 0$.

THEOREM 3.9 (Steinberg [20]). Let $h =$ order of $R_1 \dots R_n$, $r =$ number of reflections in G . Then $r = \frac{nh}{2}$.

Proof. We may label the walls of the fundamental region F so that $W_1 \dots W_s$ are mutually perpendicular, and W_{s+1}, \dots, W_n are mutually perpendicular (I.e. if the nodes corresponding to W_1, \dots, W_s are black and those corresponding to W_{s+1}, \dots, W_n are white, then each black node is linked only to white nodes and conversely). Let $E_1 = W_{s+1} \cap \dots \cap W_n$, $E_2 = W_1 \cap \dots \cap W_s$. Thus in terms of the dual basis $\{r'_i\}$, E_1 is the linear span of r'_1, \dots, r'_s and E_2 the linear span of r'_{s+1}, \dots, r'_n . Let $S = R_{s+1} \dots R_n$, $T = R_1, \dots, R_s$ and denote the orthogonal complement of E_i , $i = 1, 2$, by E_i^\perp . The restriction of S to E_1 , denoted by S_{E_1} , is the identity r_{s+1}, \dots, r_n form a basis for E_1^\perp . Since they are orthogonal to each other, $R_i r_j = 0$ for $i \neq j$, $s+1 \leq i, j \leq n$, so that $S_{E_1}^\perp = -$ identity. Similarly $T_{E_2} =$ identity, $T_{E_2}^\perp = -$ identity. We require the following

LEMMA 3.3. Let G_0 be the $n \times n$ matrix $((r_i, r_j))$ and I the $n \times n$ identity matrix. $I - G_0$ is connected. Thus, by Lemma 3.2, $I - G_0$ has a biggest positive eigenvalue λ and a corresponding eigenvector e with positive entries. Let $\sigma = \sum_{i=1}^s e_i r'_i$, $\tau = \sum_{i=s+1}^n e_i r'_i$. The plane π , determined by σ and τ , has non-trivial intersection with E_1^\perp and E_2^\perp . It follows that $S_\pi (T_\pi)$ is a reflection of π in the line through σ (τ).

Proof. The entries of $I - G_0$ are ≥ 0 , as $(r_i, r_j) \leq 0$ whenever $i \neq j$. The irreducibility of G is equivalent to saying that $I - G_0$ is connected. Let

$$G_0 = \begin{pmatrix} I & A \\ A' & I \end{pmatrix}, \quad G_0^{-1} = \begin{pmatrix} B & C \\ C' & D \end{pmatrix},$$

where A, C are $s \times n - s$ matrices (we use I to denote the identity matrix for various degrees; here degree $I = s$). The relations $r_i = \sum_{j=1}^n (r_i, r_j) r'_j$, $r'_i = \sum_{j=1}^n (r'_i, r'_j) r_j$, $1 \leq i \leq n$, show that $G_0^{-1} = ((r'_i, r'_j))$. Since $G_0^{-1} G_0 = I$, we have

$$(3.1) \quad BA + C = C' + DA' = 0$$

Let e^1 be the vector consisting of the first s components of e , e^2 the vector

¹) Geometrically, the directions of σ, τ are those in E_1, E_2 which produce the smallest angle. To prove this, one solves this minimum problem by the method of multipliers. Lagrange's equations lead to (3.2.).

consisting of the last $n - s$ components of e . The equation $(I - G_0) e = \lambda e$ becomes

$$(3.2) \quad A e^2 + \lambda e^1 = A' e^1 + \lambda e^2 = 0.$$

(3.1), (3.2) imply

$$(3.3) \quad \lambda B e^1 - C e^2 = \lambda D e^2 - C' e^1 = 0.$$

Let $\sigma = \sum_{i=1}^s e_i r'_i$, $\tau = \sum_{i=s+1}^n e_i r'_i$. (3.3) may be rewritten as

$$(3.4) \quad \begin{aligned} r'_i \cdot (\lambda \sigma - \tau) &= 0, & 1 \leq i \leq s, \\ r'_i \cdot (\lambda \tau - \sigma) &= 0, & s + 1 \leq i \leq n. \end{aligned}$$

The vectors $\lambda \sigma - \tau$, $\lambda \tau - \sigma$ are $\neq 0$ and in π . (3.4) states that $\lambda \sigma - \tau \in E_1^\perp$, $\lambda \tau - \sigma \in E_2^\perp$. Since $\sigma \in E_1$, $\sigma' = \lambda \sigma - \tau \in E_1^\perp$, we have $S(\sigma) = \sigma$, $S(\sigma') = -\sigma'$. I.e. S_π is a reflection in the line through σ . Similarly, T_π is a reflection in the line through τ .

We now return to the proof of Theorem 3.9. Let H be the subgroup generated by S, T . H_π is the group generated by S_π, T_π . Let

$$F_0 = \{v \mid v = x\sigma + y\tau, x, y > 0\} = F \cap \pi.$$

F_0 is a fundamental region for H_π . For let $\gamma \in H$, $\gamma_\pi \neq I$. Then $\gamma \neq I$ and we have $\gamma_\pi F \cap F = \gamma F \cap F \cap \pi = \Phi$. R_π is a rotation of π through twice the angle between σ and τ . We show that $\text{ord } R_\pi = h$. For let $\text{ord } R_\pi = k$. Since $R^h = I$, $R_\pi^h = I$, we have $k \leq h$. Choose $p \in F_0$. $R^k(p) = R_\pi^k(p) = p$ so that $R^k F \cap F \neq \Phi \Rightarrow R^k = I \Rightarrow h \leq k$. Thus

$h = k$. It follows that F_0 is an angular wedge of angular width $\frac{2\pi}{h}$ and

H_π is a dihedral group of order $2h$. The h transforms of σ are contained in precisely $(n-s)$ r.h.'s. The h transforms of τ are contained in precisely s r.h.'s. Every r.h. of G has a non-trivial intersection with π . Since each of the transforms of F_0 is contained in a chamber of G and each chamber is free of r.h.'s, these r.h.'s meet π only at the transforms of σ and τ . Counting the r.h.'s at the transforms of σ and τ , we obtain the count $hs + h(n-s) = hn$. Each r.h. is however counted twice, as it intersects π in a line and

thus meets two of the σ and τ transforms. Hence $r = \frac{hn}{2}$.

As a by product of the above proof, we obtain the following result required to establish Theorem 3.8.

THEOREM 3.10. $\zeta = e^{2\pi i/h}$ is an eigenvalue of R . Corresponding to ζ , we may choose an eigenvector v not lying in any r.h. (Note: if v is complex, then v is said to lie in the r.h. π iff $L(v) = 0$, $L(x) = 0$ being the equation of π).

Proof. Assume first that the R_i 's are labeled as in the proof of Theorem 3.9; i.e. the walls W_1, \dots, W_s are mutually perpendicular as are also W_{s+1}, \dots, W_n . Let π be the plane of Lemma 3.3. We choose two orthonormal vectors v_1, v_2 in π such that v_1 is not contained in any r.h. of G and

$$(3.5) \quad \begin{aligned} R(v_1) &= \cos \frac{2\pi}{h} v_1 + \sin \frac{2\pi}{h} v_2 \\ R(v_2) &= -\sin \frac{2\pi}{h} v_1 + \cos \frac{2\pi}{h} v_2 \end{aligned}$$

Let $v = v_1 - iv_2$. We conclude from (3.5) that $R(v) = e^{2i\pi/h} v$. Thus v is an eigenvector corresponding to the eigenvalue $\zeta = e^{2i\pi/h}$. v is not in any r.h. of G as v_1 is not in any r.h. of G .

For an arbitrary labeling of indices, choose a permutation i_1, \dots, i_n of $1, \dots, n$ so that the above reasoning applies to $R' = R_{i_1} \dots R_{i_n}$. By Theorem 3.7. $R = R_1 \dots R_n = \sigma R' \sigma^{-1}$ for some $\sigma \in G$. Hence $R(\sigma v) = \zeta(\sigma v)$. Since the r.h.'s are permuted by σ , we conclude that σv is also not contained in any r.h. of G .

We also require

THEOREM 3.11. 1 is not an eigenvalue of R .

REMARK. In Theorem 3.12 we obtain the characteristic equation of R , from which we may obtain Theorem 3.11. The following proof is shorter and avoids any explicit matrix representation for R .

Proof. Let π be the r.h. corresponding to the root r and σ the reflection in π . Then $v' = \sigma v$ becomes

$$(3.6) \quad v' = v - 2(v, r)r$$

Suppose that $R_1 \dots R_n v = v$, $\Leftrightarrow R_2 \dots R_n v = R_1 v$. Repeated application of (3.6) shows that $R_2 \dots R_n v = v + \lambda_2 r_2 + \dots + \lambda_n r_n$, $\lambda_2, \dots, \lambda_n$ being real numbers depending on v . Hence

$$(3.7) \quad v + \lambda_2 r_2 + \dots + \lambda_n r_n = v - 2(v, r_1)r_1$$

Since r_1, \dots, r_n are linearly independent we must have $(v, r_1) = 0 \Leftrightarrow R_1 v = v$, so that $R_2 \dots R_n v = v$. Repeating the reasoning, we con-

clude $(v, r_i) = 0, 1 \leq i \leq n, \Rightarrow v = 0$. Thus 1 is not an eigenvalue of $R_1 \dots R_n$.

We can now provide the

Proof of Theorem 3.8. Let v_1, \dots, v_n be linearly independent eigenvectors of R with v_1 chosen as in Theorem 3.10; i.e. v_1 corresponds to the eigenvalue $\zeta = e^{2i\pi/h}$ and does not lie in any r.h. of G . Let x_1, \dots, x_n be a coordinate system adapted to v_1, \dots, v_n . As $R^h = I$, all eigenvalues of R are h -th roots of I . By Theorem 3.11, 1 is not an eigenvalue of R . Hence the eigenvalues of R are $\zeta^{m_1}, \dots, \zeta^{m_n}$ where $m_1 = 1$ and $1 \leq m_1 \leq \dots \leq m_n = h - 1, 1 \leq i \leq n$. R is given by $x'_i = \zeta^{m_i} x_i, 1 \leq i \leq n$.

Let I_1, \dots, I_n be a basic set of homogeneous invariants of G of respective degrees $d_1 \leq \dots \leq d_n$. By Theorem 2.5,

$$J = \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)} \neq 0$$

off the r.h.'s of G . Hence $J \neq 0$ whenever $x = (x_1, 0, \dots, 0), x_1 \neq 0$. It follows that there exists a permutation $j = j(i)$ of 1 to n such that

$$\frac{\partial I_i}{\partial x_j}(x_1, 0, \dots, 0) \neq 0$$

for $x_1 \neq 0$ and $1 \leq i \leq n$. This means that the $x_1^{d_i-1}$ coefficient of

$$\frac{\partial I_i}{\partial x_j} \neq 0 \Rightarrow x_1^{d_i-1} x_j$$

coefficient of $I_i \neq 0, 1 \leq i \leq n$. Hence each $x_1^{d_i-1} x_j$ is invariant under R . I.e.

$$(3.8) \quad (d_i - 1) + m_j \equiv 0 \pmod{h}, 1 \leq i \leq n$$

Rewrite (3.8) as

$$(3.9) \quad d_i - 1 = (h - m_j) + \varepsilon_i h, 1 \leq i \leq n$$

where each ε_i is an integer ≥ 0 . Let $m'_j = h - m_j$. The eigenvalues of R occur in pairs, so that the set of numbers $\{m'_j\}$ is identical with $\{m_j\}$. Summing both sides of (3.9) from $i = 1$ to $i = n$, we get

$$(3.10) \quad \sum_{i=1}^n (d_i - 1) = \sum_{j=1}^n m'_j + \left(\sum_{i=1}^n \varepsilon_i \right) h$$

By Theorem 2.2, $\sum_{i=1}^n (d_i - 1) = r$. Since

$$(3.11) \quad \sum_{j=1}^n m_j' = \sum_{j=1}^n (h - m_j) = nh - \sum_{j=1}^n m_j',$$

we also have $\sum_{j=1}^n m_j' = \frac{nh}{2}$. We conclude from Theorem 3.9 that

$$\sum_{i=1}^n (d_i - 1) = \sum_{j=1}^n m_j'. \quad (3.10) \text{ shows that } \sum_{i=1}^n \varepsilon_i = 0 \Rightarrow \varepsilon_i = 0, 1 \leq i \leq n.$$

It follows from (3.9) that $d_i - 1 = m_i, 1 \leq i \leq n$.

To make effective use of Coleman's Theorem, we need the explicit expression for the characteristic equation of R .

THEOREM 3.12 (Coxeter [5], p. 218). *The characteristic equation of $R = R_1 \dots R_n$ is given by*

$$(3.12) \quad \begin{vmatrix} \frac{1 + \lambda}{2} & \lambda a_{12} & \dots & \lambda a_{1n} \\ a_{21} & \frac{1 + \lambda}{2} & \lambda a_{23} & \dots & \lambda a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{n,n-1} & \frac{1 + \lambda}{2} & \dots \end{vmatrix} = 0$$

where $a_{ij} = -\cos(\pi/p_{ij}), 1 \leq i, j \leq n$.

Proof. Let $v = \sigma v'$ where σ is a reflection in the r.h. perpendicular to the root r .

Then

$$(3.13) \quad v = v' - 2(v' \cdot r) r$$

We use (3.13) to obtain the matrix for R_j relative to the basis r'_1, \dots, r'_n .

Let $v = \sum_{i=1}^n x_i r'_i, v' = \sum_{i=1}^n x'_i r'_i$. Then $v' \cdot r_j = x'_j, r_j = \sum_{i=1}^n a_{ij} r'_i$.

Substituting into (3.13), we get

$$(3.14) \quad v = R_j v' \Leftrightarrow x_i = x'_i - 2a_{ij} x'_j, 1 \leq i \leq n$$

Let

$$v = R_1 v^{(1)}, v^{(1)} = R_2 v^{(2)}, \dots, v^{(n-1)} = R_n v^{(n)}$$

