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degrees. The second method (Theorem 3.14) is valid for an arbitrary field of characteristic 0, but is less effective than the first in the real case.

We first prove that the degrees of the basic invariants are independent of any particular basis.

**THEOREM 3.1.** *Let  $G$  a finite reflection group acting on the  $n$ -dimensional vector space  $V$ . Let  $I_1, \dots, I_n$  be homogeneous polynomials of respective degrees  $d_1 \leq \dots \leq d_n$  forming a basis for the invariants of  $G$ .  $d_1, \dots, d_n$  are independent of the chosen basis  $I_1, \dots, I_n$ .*

*Proof.* Let  $J_1, \dots, J_n$  be another set of homogeneous invariants forming a basis for the invariants of  $G$ . Let  $d'_1 \leq \dots \leq d'_n$  be the respective degrees of  $J_1, \dots, J_n$ . We must show that  $d'_i = d_i, 1 \leq i \leq n$ . If not, then let  $i_0$  be the smallest  $i$  such that  $d'_{i_0} \neq d_{i_0}$ , say  $d'_{i_0} < d_{i_0}$ . Each  $J_i$  is a polynomial in those  $I_i$ 's whose degree  $\leq \deg J_i$ . It follows that for  $1 \leq i \leq i_0$ ,  $J_i = P_i(I_1, \dots, I_{i_0-1})$ ,  $P_i(y_1, \dots, y_{i_0-1})$  being a polynomial in  $y_1, \dots, y_{i_0-1}$ . Hence  $J_1, \dots, J_{i_0}$  are algebraically dependent over  $k$  ([22], Vol. 1, p. 181), contradicting that  $J_1, \dots, J_n$  are algebraically independent over  $k$  (Theorem 1.2). Thus  $d'_i = d_i, 1 \leq i \leq n$ .

Theorem 3.1. shows that the numbers  $d_1, \dots, d_n$  are determined by  $G$ . We shall give an effective method for the computation of the  $d_i$ 's in case the underlying field  $k$  is real. We first digress to discuss the classification of the finite real reflection groups.

### 1. THE CLASSIFICATION OF THE FINITE REAL REFLECTION GROUPS

These groups have been classified by Coxeter [6]. We give here a brief description of the theory, as we require it for the computation of the  $d_i$ 's.

We first observe that we may assume  $G$  to be orthogonal.

**THEOREM 3.2.** *Let  $G$  be a finite group acting on the  $n$ -dimensional Euclidean space  $R^n$ . There exists a non-singular transformation  $\tau$  on  $R^n$  such that the group  $\tau^{-1} G \tau$  consists of orthogonal transformations.*

*Proof.* Let  $P(x) = \sum_{\sigma \in G} (\sigma x, \sigma x)$  where  $x = (x_1, \dots, x_n)$  and  $(x, y)$  is the inner product of  $x$  and  $y$ . For  $x \neq 0$ , each  $(\sigma x, \sigma x) > 0$  so that  $P(x) > 0$ . Furthermore for  $\sigma_1 \in G, P(\sigma_1 x) = \sum_{\sigma \in G} (\sigma \sigma_1 x, \sigma \sigma_1 x) = \sum_{\sigma \in G} (\sigma x, \sigma x) = P(x)$ . Thus  $P(x)$  is a positive definite quadratic form

invariant under  $G$ . Choose  $x = \tau y$  so that  $P(\tau y) = (y, y)$ . We have  $(\tau^{-1}\sigma\tau y, \tau^{-1}\sigma\tau y) = P(\sigma\tau y) = P(\tau y) = (y, y)$ ,  $\sigma \in G$ , so that the transformations  $\tau^{-1}\sigma\tau$  are orthogonal.

Thus all transformations of  $G$  become orthogonal after a suitable linear change of variables. We assume from now on that  $G$  is orthogonal. If  $G$  is a finite reflection group, this condition is equivalent to demanding that all reflections of  $G$  are orthogonal. I.e. for any reflection  $\sigma$ ,  $\sigma$  fixes all vectors in the r.h.  $\pi$  and  $\sigma(v) = -v$ , iff  $v$  is perpendicular to  $\pi$ . The two unit vectors perpendicular to  $\pi$  are called roots of  $G$ . The set of all roots is called the root system of  $G$ .

DEFINITION 3.1. Let  $F$  be a region of  $R^n$ ,  $G$  a finite group acting on  $R^n$ .  $F$  is a fundamental region for  $G$  iff:

- i)  $\sigma_1 F \cap \sigma_2 F = \Phi$  whenever  $\sigma_1 \neq \sigma_2$ ,
- ii)  $R^n = \bigcup_{\sigma \in G} \sigma \bar{F}$ ,  $\bar{F}$  being the closure of  $F$ .

We remark that it suffices to know i) for  $\sigma_1 = e$ , the identity of  $G$ . For  $\sigma_1 F \cap \sigma_2 F = \Phi$  iff  $\sigma_1^{-1}(\sigma_1 F \cap \sigma_2 F) = F \cap \sigma_1^{-1}\sigma_2 F = \Phi$ . If  $F$  is a fundamental region, then so is  $\sigma F$ ,  $\sigma \in G$ . The group  $G$  permutes these fundamental regions and acts transitively on them.

THEOREM 3.3. Let  $G$  be a finite reflection group acting on  $R^n$ . Assume that the roots of  $G$  span  $R^n$  ( $G$  is then called a Coxeter group). The complement of the union of the r.h.'s of  $G$  consist of  $|G|$  fundamental regions called the chambers of  $G$ .  $G$  permutes these chambers and acts transitively on them. Each chamber  $F$  is bounded by  $n$  r.h.'s called the walls of  $F$ . Let  $r_1, \dots, r_n$  be the  $n$  roots perpendicular to the  $n$  walls  $W_1, \dots, W_n$  and pointing into  $F$ , and let  $R_i$  be the reflection in  $W_i$ . The  $r_i$ 's are linearly independent and  $r_i \cdot r_j = -\cos \pi/p_{ij}$ ,  $p_{ii} = 1$  and  $p_{ij}$  being an integer  $\geq 2$  if  $i \neq j$ . The  $R_i$ 's generate  $G$ .

We have  $F = \{x \mid x \cdot r_i > 0, 1 \leq i \leq n\}$ .  $F$  may also be described as follows. Choose  $\{r'_1, \dots, r'_n\}$  to be the dual basis to  $\{r_1, \dots, r_n\}$ ; i.e.

$(r_i, r_j) = \delta_{ij}$ . For any  $x$ ,  $x = \sum_{i=1}^n (x \cdot r_i) r'_i$ . Thus

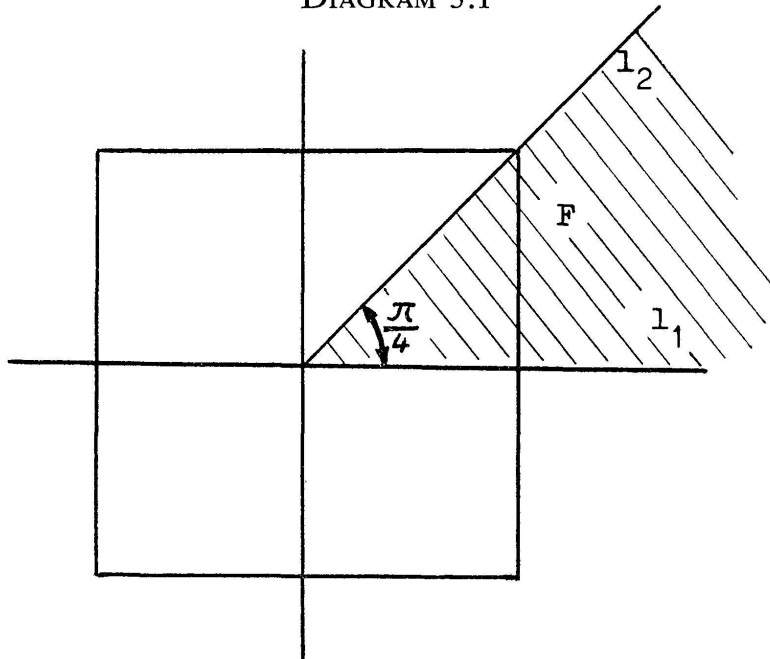
$$F = \{x \mid x = \sum_{i=1}^n \lambda_i r'_i, \lambda_i > 0 \text{ for } 1 \leq i \leq n\}.$$

$F$  is thus a wedge with  $n$  walls, the vectors  $r'_i$  lying along its edges. The angle between the walls  $W_i, W_j$  ( $i \neq j$ ) is readily seen to be  $\pi/p_{ij}$ . We refer

to  $\{r_1, \dots, r_n\}$  as a fundamental system of roots and to  $R_1, \dots, R_n$  as a fundamental system of reflections.

As a simple illustration of the above concepts, we choose  $G$  to be the group of symmetries of a regular  $n$ -gon  $p_n$ .  $G$  is then called the dihedral group of order  $2n$  and we denote it by  $H_2^n$ . Assume that the center of the polygon is at the origin. We choose in this case two rays  $l_1, l_2$  emanating from the origin making an angle  $\pi/n$ , one of the rays passing through a vertex of  $p_n$ , the other through a mid-point of a side of  $p_n$  (see the diagram where  $n = 4$ ).  $F$  is the wedge with sides  $l_1, l_2$ . The reflections in  $l_1, l_2$  generate  $H_2^n$ .

DIAGRAM 3.1



For any Coxeter group  $G$  acting on  $R^n$ , we introduce the associated Coxeter graph  $\mathcal{G}$  as follows. Let  $\mathcal{G}$  consist of  $n$  points, called the nodes and label these as  $1, \dots, n$ . We set up the 1 - 1 correspondence  $i \leftrightarrow r_i$ ,  $r_1, \dots, r_n$  being the fundamental root system of Theorem 3.3. The  $i$ -th and  $j$ -th node ( $i \neq j$ ) are joined by a branch iff  $(r_i, r_j) \neq 0$ . If this be the case then  $p_{ij} \geq 3$ ; we mark the branch joining  $i$  to  $j$  by  $p_{ij}$  whenever  $p_{ij} > 3$ , and omit a mark if  $p_{ij} = 3$ . Eg. the graph associated with  $H_2^n$  is  $\circ - \circ$  for  $n = 3$  and  $\circ - \overset{n}{-} \circ$  for  $n \geq 4$ .

The motivation for the rather artificial looking definition of  $\mathcal{G}$  stems from the following facts.

**THEOREM 3.4.** *Let  $G$  be a Coxeter group acting on  $R^n$ .  $G$  is irreducible iff its corresponding graph is connected.*

*Proof.* If the graph of  $G$  has more than one component, then the root system  $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$  where  $\mathcal{R}_1, \mathcal{R}_2$  are disjoint and non-empty, the roots

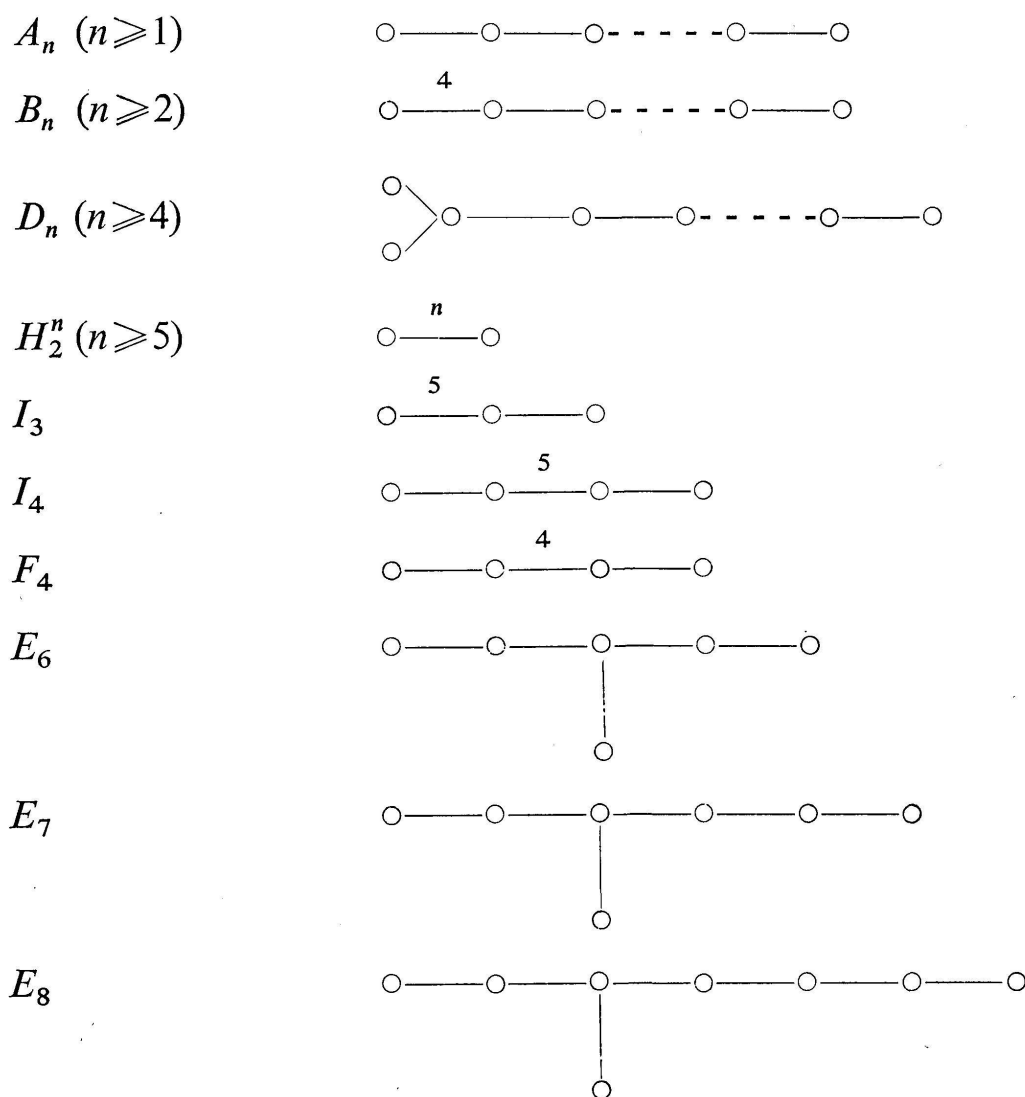
in  $\mathcal{R}_1$  being perpendicular to those in  $\mathcal{R}_2$ . Let  $V$  be the span of the roots in  $\mathcal{R}_1$ . If  $\sigma$  is a reflection corresponding to a root in  $\mathcal{R}_1$ , then  $\sigma|_V$  is a reflection of  $V$ . If  $\sigma$  is a reflection corresponding to a root in  $\mathcal{R}_2$ , then  $\sigma|_V = \text{identity}$ . Since the reflections generate  $G$ ,  $V$  is a proper invariant subspace.

Conversely, let  $V$  be a proper invariant subspace of  $G$ . Then so is the orthogonal complement  $V^\perp$ . The proof of Theorem 2.7 shows that every root is either in  $V$  or  $V^\perp$ . Since the roots span  $R^n$ , there are roots both in  $V$  and  $V^\perp$ . Since the roots in  $\mathcal{R} \cap V$  are perpendicular to those of  $\mathcal{R} \cap V^\perp$ , the graph of  $G$  consists of at least two components.

Coxeter has found all graphs corresponding to the irreducible Coxeter groups. We have the following classification.

**THEOREM 3.5.** *Let  $\mathcal{G}$  be a connected Coxeter graph. The following list exhausts the possibilities for  $\mathcal{G}$ .*

DIAGRAM 3.2



In each case the subscript denotes the number of nodes. The above list yields all irreducible Coxeter groups up to conjugacy. I.e. two irreducible groups which are conjugate subgroups of the orthogonal group have the same graph and conversely.

We give a brief description of the groups listed above.

$A_n$ . Let  $S_{n+1}$  be the symmetric group of linear transformations  $x'_i = x_{\sigma(i)}$ ,  $1 \leq i \leq n+1$ ,  $\sigma(i)$  being any permutation of  $1, \dots, n+1$ . Let  $V = \{x \mid x_1 + \dots + x_{n+1} = 0\}$  and  $A_n = S_{n+1} |_{V}$ .  $A_n$  is the group of symmetries of the regular  $n$ -simplex whose vertices are the permutations of  $(-1, \dots, -1, n)$ .

$B_n$  is the group of symmetries of the  $n$  cube with vertices  $(\pm 1, \dots, \pm 1)$ . It consists of the  $2^n n!$  linear transformations  $x'_i = \pm x_{\sigma(i)}$ ,  $1 \leq i \leq n$ , the  $\pm$  signs being chosen independently and  $\sigma(i)$  an arbitrary permutation of  $1, \dots, n$ .

$D_n$  consists of the  $2^{n-1} n!$  linear transformations  $x'_i = \pm x_{\sigma(i)}$ ,  $1 \leq i \leq n$ , where  $\sigma(i)$  is any permutation of  $1, \dots, n$  and the number of  $-$  signs is even. It is readily checked that  $D_n$  is a subgroup of index 2 in  $B_n$ .

$H_2^n$  is the dihedral group of  $2n$  symmetries of the regular  $n$ -gon.

$I_3$  is the icosahedral group, i.e. the group of symmetries of the icosahedron.

$I_4, F_4$  are the groups of symmetries of certain 4-dimensional regular polytopes described in ([5], p. 156)

$E_6, E_7, E_8$  are the groups of symmetries of certain polytopes in  $R^6, R^7, R^8$  known as Gosset's figures and described in ([5], p. 202)

An inspection of diagram 3.2 reveals that the graphs are of two types, those consisting of one chain and those consisting of three chains joined at a node. We refer to these graphs and their associated groups as being of types I and II. It can be shown that the groups of type I are precisely those which are the groups of symmetries of the regular polytopes ([5], p. 199).

The following theorem gives a complete description of all finite reflection groups acting on  $R^n$ .

**THEOREM 3.6.** *Let  $G$  be a finite reflection group acting on  $R^n$ .  $R^n$  is a direct sum of mutually orthogonal subspaces  $V_0, V_1, \dots, V_k$  with the following properties.*

- 1) *Let  $G_i = G|_{V_i}$  = the restrictions of the elements of  $G$  to  $V_i$ . Then  $G$  is isomorphic to  $G_0 \times G_1 \times \dots \times G_k$ .*
- 2)  *$G_0$  consists only of the identity transformation on  $V_0$ .*

- 3) Each  $G_i$ ,  $1 \leq i \leq k$ , is one of the groups described in Theorem 3.5.  $G$  is a Coxeter group iff  $V_0 = 0$ .

The proof of Theorem 3.6 is identical with that of Theorem 2.7. We simply observe that we may now choose the  $V_i$ 's to be mutually orthogonal.

## 2. THE COMPUTATION OF THE DEGREES FOR REAL FINITE REFLECTION GROUPS

Let  $G$  be a finite irreducible orthogonal reflection group acting on the  $n$ -dimensional Euclidean space  $R^n$ . Let  $F$  be a fundamental region as described in Theorem 3.3 and  $R_1, \dots, R_n$  the  $n$  reflections in the walls of  $F$ . We shall relate the degrees  $d_1, \dots, d_n$  of the basic homogeneous invariants to the eigenvalues of  $R_1 \dots R_n$ . We first prove

**THEOREM 3.7.** *Let  $\sigma(i)$  be any permutation of  $1, \dots, n$ . Then  $R_1 \dots R_n$  is conjugate to  $R_{\sigma(1)} \dots R_{\sigma(n)}$*

*Proof.* Observe that  $R_1 (R_1 \dots R_n) R_1 = R_2 \dots R_n R_1$  so that all cyclic permutations yield conjugate transformations. We may also permute any two adjacent  $R_i$ 's for which the corresponding walls are orthogonal, as the  $R_i$ 's then commute. Theorem 3.7 will then follow from the following

**LEMMA 3.1.** Let  $p_1, \dots, p_n$  be nodes of a tree  $T$ . Any circular arrangement of  $1, \dots, n$  can be obtained from a sequence of interchanges of pairs  $i, j$  which are adjacent on the circle and for which  $p_i, p_j$  are not linked in  $T$ .

*Proof of Lemma 3.1.* We proceed by induction, the result being obvious for  $n = 1$  or  $2$ . We may assume that  $p_n$  is an end node of the tree, i.e. it links to precisely one other node. We first rearrange  $1, \dots, n - 1$  as we wish. To show that this can be done, we just consider the possibility  $---inj---$  where  $p_i, p_j$  are not linked. If  $p_i, p_n$  are not linked, then we interchange first  $i, n$  and then  $i, j$ , obtaining  $---nji---$ . If  $p_j, p_n$  are not linked, then we first interchange  $j, n$  and then  $j, i$ , obtaining  $---jin---$ . We may therefore arrange  $1, \dots, n - 1$  in the desired order. Shifting  $n$  in one direction, which is permissible as  $n$  just fails to commute with one element, we obtain the desired arrangement of  $1, \dots, n$ .

In view of Theorem 3.7, the eigenvalues of  $R_1 \dots R_n$  are independent of the order in which the  $R_i$ 's appear. They are also independent of the particularly chosen  $F$ . For let  $F'$  be another fundamental region as described in Theorem 3.3. Then  $F' = \sigma F$ ,  $\sigma \in G$ . The reflections in the walls of  $F'$