

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 24 (1978)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** INVARIANTS OF FINITE REFLECTION GROUPS  
**Autor:** Flatto, Leopold  
**Kapitel:** CHAPTER III THE DEGREES OF THE BASIC INVARIANTS  
**DOI:** <https://doi.org/10.5169/seals-49704>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 07.02.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

If  $V_i \not\subset \pi$ , then  $V_i + \pi = V$  and we conclude from (2.22) that  $\dim V_i = \dim (V_i \cap \pi) + 1$ . I.e.  $V_i \cap \pi$  is a hyperplane in  $V_i$  and  $\sigma|_{V_i}$  a reflection on  $V_i$ . Choose  $u \in V_i - \pi$  so that  $u$  is an eigenvector of  $\sigma$ .  $u$  is a multiple of the root  $v$ , so that  $v \in V_i$ . Thus  $\sigma|_{V_i}$  is a reflection of  $V_i$  if  $v \in V_i$ , and the identity if  $v \notin V_i$ . Furthermore, each root  $v$  is in some  $V_i$ ,  $r + 1 \leq i \leq s$ , otherwise the corresponding reflection  $\sigma$  would have been the identity.

Let  $\tilde{G}_i =$  subgroup generated by those reflections whose roots are in  $V_i$ ,  $1 \leq i \leq s$ . It is readily checked that  $G = \tilde{G}_1 \times \dots \times \tilde{G}_s$ ,  $G_i = \tilde{G}_i|_{V_i}$ . If  $\sigma \in \tilde{G}_i$  and  $\sigma|_{V_i} =$  identity then  $\sigma =$  identity. The mapping  $\sigma \rightarrow \sigma|_{V_i}$  is thus an isomorphism from  $\tilde{G}_i$  onto  $G_i$ .

**THEOREM 2.8.** *Let  $G$  be a finite reflection group acting on  $V$  and decompose  $V$  as in Theorem 2.7. Every polynomial invariant under  $G$  is a polynomial in the invariant polynomials of  $G_1, \dots, G_s$ .*

*Proof.* For each  $v \in V$ , write  $v = v_1 + \dots + v_s$ ,  $v_i \in V_i$ . By Theorem 2.7, for each  $\sigma \in G$ , we may write  $\sigma v = \sigma_1 v_1 + \dots + \sigma_s v_s$ ,  $\sigma_i \in G_i$ . For any polynomial function  $p(v)$  on  $V$ , we have  $p(v) = \sum_{i=1}^N p_{i1}(v_1) \dots p_{is}(v_s)$  where  $p_{ij}(v_j)$  is a polynomial function on  $V_j$ . If  $p(v)$  is invariant under  $G$ , then

$$(2.23) \quad p(v) = \frac{1}{|G|} \sum_{\sigma \in G} p(\sigma v) = \sum_{i=1}^N I_{i1}(v_1) \dots I_{is}(v_s)$$

where

$$(2.24) \quad I_{ij}(v_j) = \frac{1}{|G_j|} \sum_{\sigma_j \in G_j} p_{ij}(\sigma_j v_j)$$

is an invariant of  $G_j$ .

## CHAPTER III

### THE DEGREES OF THE BASIC INVARIANTS

We determine the degrees of the basic homogeneous invariants in case  $G$  is a finite reflection group. We present two different methods. The first one (Theorem 3.8), restricts itself to the case where  $k$  is the real field and has the advantage of providing an effective method for computing the

degrees. The second method (Theorem 3.14) is valid for an arbitrary field of characteristic 0, but is less effective than the first in the real case.

We first prove that the degrees of the basic invariants are independent of any particular basis.

**THEOREM 3.1.** *Let  $G$  a finite reflection group acting on the  $n$ -dimensional vector space  $V$ . Let  $I_1, \dots, I_n$  be homogeneous polynomials of respective degrees  $d_1 \leq \dots \leq d_n$  forming a basis for the invariants of  $G$ .  $d_1, \dots, d_n$  are independent of the chosen basis  $I_1, \dots, I_n$ .*

*Proof.* Let  $J_1, \dots, J_n$  be another set of homogeneous invariants forming a basis for the invariants of  $G$ . Let  $d'_1 \leq \dots \leq d'_n$  be the respective degrees of  $J_1, \dots, J_n$ . We must show that  $d'_i = d_i$ ,  $1 \leq i \leq n$ . If not, then let  $i_0$  be the smallest  $i$  such that  $d'_{i_0} \neq d_{i_0}$ , say  $d'_{i_0} < d_{i_0}$ . Each  $J_i$  is a polynomial in those  $I_i$ 's whose degree  $\leq \deg J_i$ . It follows that for  $1 \leq i \leq i_0$ ,  $J_i = P_i(I_1, \dots, I_{i_0-1})$ ,  $P_i(y_1, \dots, y_{i_0-1})$  being a polynomial in  $y_1, \dots, y_{i_0-1}$ . Hence  $J_1, \dots, J_{i_0}$  are algebraically dependent over  $k$  ([22], Vol. 1, p. 181), contradicting that  $J_1, \dots, J_n$  are algebraically independent over  $k$  (Theorem 1.2). Thus  $d'_i = d_i$ ,  $1 \leq i \leq n$ .

Theorem 3.1. shows that the numbers  $d_1, \dots, d_n$  are determined by  $G$ . We shall give an effective method for the computation of the  $d_i$ 's in case the underlying field  $k$  is real. We first digress to discuss the classification of the finite real reflection groups.

## 1. THE CLASSIFICATION OF THE FINITE REAL REFLECTION GROUPS

These groups have been classified by Coxeter [6]. We give here a brief description of the theory, as we require it for the computation of the  $d_i$ 's.

We first observe that we may assume  $G$  to be orthogonal.

**THEOREM 3.2.** *Let  $G$  be a finite group acting on the  $n$ -dimensional Euclidean space  $R^n$ . There exists a non-singular transformation  $\tau$  on  $R^n$  such that the group  $\tau^{-1} G \tau$  consists of orthogonal transformations.*

*Proof.* Let  $P(x) = \sum_{\sigma \in G} (\sigma x, \sigma x)$  where  $x = (x_1, \dots, x_n)$  and  $(x, y)$  is the inner product of  $x$  and  $y$ . For  $x \neq 0$ , each  $(\sigma x, \sigma x) > 0$  so that  $P(x) > 0$ . Furthermore for  $\sigma_1 \in G$ ,  $P(\sigma_1 x) = \sum_{\sigma \in G} (\sigma \sigma_1 x, \sigma \sigma_1 x) = \sum_{\sigma \in G} (\sigma x, \sigma x) = P(x)$ . Thus  $P(x)$  is a positive definite quadratic form

invariant under  $G$ . Choose  $x = \tau y$  so that  $P(\tau y) = (y, y)$ . We have  $(\tau^{-1}\sigma\tau y, \tau^{-1}\sigma\tau y) = P(\sigma\tau y) = P(\tau y) = (y, y)$ ,  $\sigma \in G$ , so that the transformations  $\tau^{-1}\sigma\tau$  are orthogonal.

Thus all transformations of  $G$  become orthogonal after a suitable linear change of variables. We assume from now on that  $G$  is orthogonal. If  $G$  is a finite reflection group, this condition is equivalent to demanding that all reflections of  $G$  are orthogonal. I.e. for any reflection  $\sigma$ ,  $\sigma$  fixes all vectors in the r.h.  $\pi$  and  $\sigma(v) = -v$ , iff  $v$  is perpendicular to  $\pi$ . The two unit vectors perpendicular to  $\pi$  are called roots of  $G$ . The set of all roots is called the root system of  $G$ .

DEFINITION 3.1. Let  $F$  be a region of  $R^n$ ,  $G$  a finite group acting on  $R^n$ .  $F$  is a fundamental region for  $G$  iff:

- i)  $\sigma_1 F \cap \sigma_2 F = \Phi$  whenever  $\sigma_1 \neq \sigma_2$ ,
- ii)  $R^n = \bigcup_{\sigma \in G} \sigma \bar{F}$ ,  $\bar{F}$  being the closure of  $F$ .

We remark that it suffices to know i) for  $\sigma_1 = e$ , the identity of  $G$ . For  $\sigma_1 F \cap \sigma_2 F = \Phi$  iff  $\sigma_1^{-1}(\sigma_1 F \cap \sigma_2 F) = F \cap \sigma_1^{-1}\sigma_2 F = \Phi$ . If  $F$  is a fundamental region, then so is  $\sigma F$ ,  $\sigma \in G$ . The group  $G$  permutes these fundamental regions and acts transitively on them.

THEOREM 3.3. Let  $G$  be a finite reflection group acting on  $R^n$ . Assume that the roots of  $G$  span  $R^n$  ( $G$  is then called a Coxeter group). The complement of the union of the r.h.'s of  $G$  consist of  $|G|$  fundamental regions called the chambers of  $G$ .  $G$  permutes these chambers and acts transitively on them. Each chamber  $F$  is bounded by  $n$  r.h.'s called the walls of  $F$ . Let  $r_1, \dots, r_n$  be the  $n$  roots perpendicular to the  $n$  walls  $W_1, \dots, W_n$  and pointing into  $F$ , and let  $R_i$  be the reflection in  $W_i$ . The  $r_i$ 's are linearly independent and  $r_i \cdot r_j = -\cos \pi/p_{ij}$ ,  $p_{ii} = 1$  and  $p_{ij}$  being an integer  $\geq 2$  if  $i \neq j$ . The  $R_i$ 's generate  $G$ .

We have  $F = \{x \mid x \cdot r_i > 0, 1 \leq i \leq n\}$ .  $F$  may also be described as follows. Choose  $\{r'_1, \dots, r'_n\}$  to be the dual basis to  $\{r_1, \dots, r_n\}$ ; i.e.  $(r_i, r_j) = \delta_{ij}$ . For any  $x$ ,  $x = \sum_{i=1}^n (x \cdot r_i) r'_i$ . Thus

$$F = \{x \mid x = \sum_{i=1}^n \lambda_i r'_i, \lambda_i > 0 \text{ for } 1 \leq i \leq n\}.$$

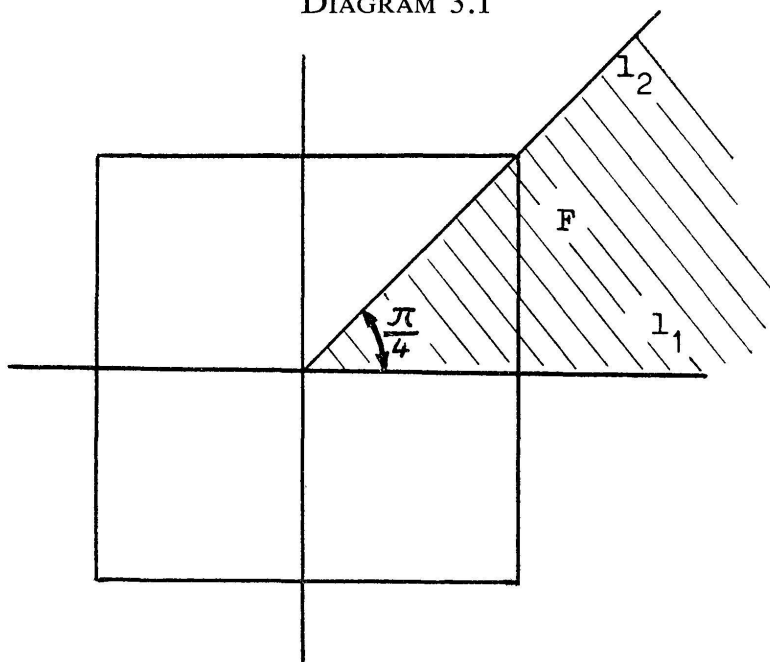
$F$  is thus a wedge with  $n$  walls, the vectors  $r'_i$  lying along its edges. The angle between the walls  $W_i, W_j$  ( $i \neq j$ ) is readily seen to be  $\pi/p_{ij}$ . We refer



to  $\{r_1, \dots, r_n\}$  as a fundamental system of roots and to  $R_1, \dots, R_n$  as a fundamental system of reflections.

As a simple illustration of the above concepts, we choose  $G$  to be the group of symmetries of a regular  $n$ -gon  $p_n$ .  $G$  is then called the dihedral group of order  $2n$  and we denote it by  $H_2^n$ . Assume that the center of the polygon is at the origin. We choose in this case two rays  $l_1, l_2$  emanating from the origin making an angle  $\pi/n$ , one of the rays passing through a vertex of  $p_n$ , the other through a mid-point of a side of  $p_n$  (see the diagram where  $n = 4$ ).  $F$  is the wedge with sides  $l_1, l_2$ . The reflections in  $l_1, l_2$  generate  $H_2^n$ .

DIAGRAM 3.1



For any Coxeter group  $G$  acting on  $R^n$ , we introduce the associated Coxeter graph  $\mathcal{G}$  as follows. Let  $\mathcal{G}$  consist of  $n$  points, called the nodes and label these as  $1, \dots, n$ . We set up the 1 - 1 correspondence  $i \leftrightarrow r_i$ ,  $r_1, \dots, r_n$  being the fundamental root system of Theorem 3.3. The  $i$ -th and  $j$ -th node ( $i \neq j$ ) are joined by a branch iff  $(r_i, r_j) \neq 0$ . If this be the case then  $p_{ij} \geq 3$ ; we mark the branch joining  $i$  to  $j$  by  $p_{ij}$  whenever  $p_{ij} > 3$ , and omit a mark if  $p_{ij} = 3$ . Eg. the graph associated with  $H_2^n$  is  $\circ \text{---} \circ$  for  $n = 3$  and  $\circ \text{---}^n \circ$  for  $n \geq 4$ .

The motivation for the rather artificial looking definition of  $\mathcal{G}$  stems from the following facts.

**THEOREM 3.4.** *Let  $G$  be a Coxeter group acting on  $R^n$ .  $G$  is irreducible iff its corresponding graph is connected.*

*Proof.* If the graph of  $G$  has more than one component, then the root system  $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$  where  $\mathcal{R}_1, \mathcal{R}_2$  are disjoint and non-empty, the roots

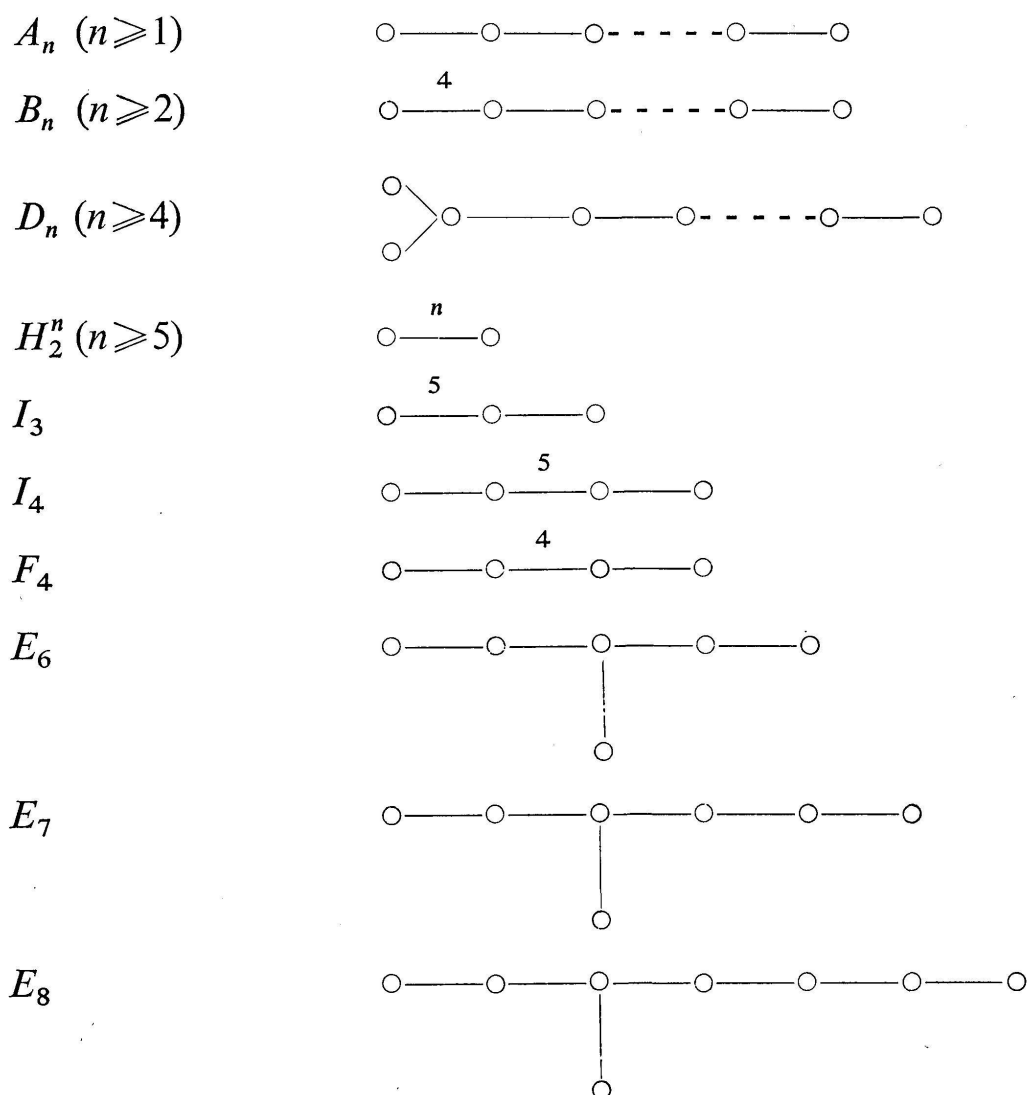
in  $\mathcal{R}_1$  being perpendicular to those in  $\mathcal{R}_2$ . Let  $V$  be the span of the roots in  $\mathcal{R}_1$ . If  $\sigma$  is a reflection corresponding to a root in  $\mathcal{R}_1$ , then  $\sigma|_V$  is a reflection of  $V$ . If  $\sigma$  is a reflection corresponding to a root in  $\mathcal{R}_2$ , then  $\sigma|_V = \text{identity}$ . Since the reflections generate  $G$ ,  $V$  is a proper invariant subspace.

Conversely, let  $V$  be a proper invariant subspace of  $G$ . Then so is the orthogonal complement  $V^\perp$ . The proof of Theorem 2.7 shows that every root is either in  $V$  or  $V^\perp$ . Since the roots span  $R^n$ , there are roots both in  $V$  and  $V^\perp$ . Since the roots in  $\mathcal{R} \cap V$  are perpendicular to those of  $\mathcal{R} \cap V^\perp$ , the graph of  $G$  consists of at least two components.

Coxeter has found all graphs corresponding to the irreducible Coxeter groups. We have the following classification.

**THEOREM 3.5.** *Let  $\mathcal{G}$  be a connected Coxeter graph. The following list exhausts the possibilities for  $\mathcal{G}$ .*

DIAGRAM 3.2



In each case the subscript denotes the number of nodes. The above list yields all irreducible Coxeter groups up to conjugacy. I.e. two irreducible groups which are conjugate subgroups of the orthogonal group have the same graph and conversely.

We give a brief description of the groups listed above.

$A_n$ . Let  $S_{n+1}$  be the symmetric group of linear transformations  $x'_i = x_{\sigma(i)}$ ,  $1 \leq i \leq n+1$ ,  $\sigma(i)$  being any permutation of  $1, \dots, n+1$ . Let  $V = \{x \mid x_1 + \dots + x_{n+1} = 0\}$  and  $A_n = S_{n+1}|_V$ .  $A_n$  is the group of symmetries of the regular  $n$ -simplex whose vertices are the permutations of  $(-1, \dots, -1, n)$ .

$B_n$  is the group of symmetries of the  $n$  cube with vertices  $(\pm 1, \dots, \pm 1)$ . It consists of the  $2^n n!$  linear transformations  $x'_i = \pm x_{\sigma(i)}$ ,  $1 \leq i \leq n$ , the  $\pm$  signs being chosen independently and  $\sigma(i)$  an arbitrary permutation of  $1, \dots, n$ .

$D_n$  consists of the  $2^{n-1} n!$  linear transformations  $x'_i = \pm x_{\sigma(i)}$ ,  $1 \leq i \leq n$ , where  $\sigma(i)$  is any permutation of  $1, \dots, n$  and the number of  $-$  signs is even. It is readily checked that  $D_n$  is a subgroup of index 2 in  $B_n$ .

$H_2^n$  is the dihedral group of  $2n$  symmetries of the regular  $n$ -gon.

$I_3$  is the icosahedral group, i.e. the group of symmetries of the icosahedron.

$I_4, F_4$  are the groups of symmetries of certain 4-dimensional regular polytopes described in ([5], p. 156)

$E_6, E_7, E_8$  are the groups of symmetries of certain polytopes in  $R^6, R^7, R^8$  known as Gosset's figures and described in ([5], p. 202)

An inspection of diagram 3.2 reveals that the graphs are of two types, those consisting of one chain and those consisting of three chains joined at a node. We refer to these graphs and their associated groups as being of types I and II. It can be shown that the groups of type I are precisely those which are the groups of symmetries of the regular polytopes ([5], p. 199).

The following theorem gives a complete description of all finite reflection groups acting on  $R^n$ .

**THEOREM 3.6.** *Let  $G$  be a finite reflection group acting on  $R^n$ .  $R^n$  is a direct sum of mutually orthogonal subspaces  $V_0, V_1, \dots, V_k$  with the following properties.*

- 1) *Let  $G_i = G|_{V_i}$  = the restrictions of the elements of  $G$  to  $V_i$ . Then  $G$  is isomorphic to  $G_0 \times G_1 \times \dots \times G_k$ .*
- 2)  *$G_0$  consists only of the identity transformation on  $V_0$ .*

- 3) Each  $G_i$ ,  $1 \leq i \leq k$ , is one of the groups described in Theorem 3.5.  
 $G$  is a Coxeter group iff  $V_0 = 0$ .

The proof of Theorem 3.6 is identical with that of Theorem 2.7. We simply observe that we may now choose the  $V_i$ 's to be mutually orthogonal.

## 2. THE COMPUTATION OF THE DEGREES FOR REAL FINITE REFLECTION GROUPS

Let  $G$  be a finite irreducible orthogonal reflection group acting on the  $n$ -dimensional Euclidean space  $R^n$ . Let  $F$  be a fundamental region as described in Theorem 3.3 and  $R_1, \dots, R_n$  the  $n$  reflections in the walls of  $F$ . We shall relate the degrees  $d_1, \dots, d_n$  of the basic homogeneous invariants to the eigenvalues of  $R_1 \dots R_n$ . We first prove

**THEOREM 3.7.** *Let  $\sigma(i)$  be any permutation of  $1, \dots, n$ . Then  $R_1 \dots R_n$  is conjugate to  $R_{\sigma(1)} \dots R_{\sigma(n)}$*

*Proof.* Observe that  $R_1 (R_1 \dots R_n) R_1 = R_2 \dots R_n R_1$  so that all cyclic permutations yield conjugate transformations. We may also permute any two adjacent  $R_i$ 's for which the corresponding walls are orthogonal, as the  $R_i$ 's then commute. Theorem 3.7 will then follow from the following

**LEMMA 3.1.** Let  $p_1, \dots, p_n$  be nodes of a tree  $T$ . Any circular arrangement of  $1, \dots, n$  can be obtained from a sequence of interchanges of pairs  $i, j$  which are adjacent on the circle and for which  $p_i, p_j$  are not linked in  $T$ .

*Proof of Lemma 3.1.* We proceed by induction, the result being obvious for  $n = 1$  or  $2$ . We may assume that  $p_n$  is an end node of the tree, i.e. it links to precisely one other node. We first rearrange  $1, \dots, n-1$  as we wish. To show that this can be done, we just consider the possibility  $---inj---$  where  $p_i, p_j$  are not linked. If  $p_i, p_n$  are not linked, then we interchange first  $i, n$  and then  $i, j$ , obtaining  $---nji---$ . If  $p_j, p_n$  are not linked, then we first interchange  $j, n$  and then  $j, i$ , obtaining  $---jin---$ . We may therefore arrange  $1, \dots, n-1$  in the desired order. Shifting  $n$  in one direction, which is permissible as  $n$  just fails to commute with one element, we obtain the desired arrangement of  $1, \dots, n$ .

In view of Theorem 3.7, the eigenvalues of  $R_1 \dots R_n$  are independent of the order in which the  $R_i$ 's appear. They are also independent of the particularly chosen  $F$ . For let  $F'$  be another fundamental region as described in Theorem 3.3. Then  $F' = \sigma F$ ,  $\sigma \in G$ . The reflections in the walls of  $F'$

are given by  $R'_i = \sigma R_i \sigma^{-1}$ ,  $1 \leq i \leq n$ , so that  $R'_1 \dots R'_n = \sigma R_1 \dots R_n \sigma^{-1}$ . The main result of the present section is the following

**THEOREM 3.8** (Coleman [8]). *Let  $R_1 \dots R_n$  have order  $h$ . Let  $\zeta = e^{2\pi i/h}$ . The eigenvalues of  $R_1 \dots R_n$  are given by  $\zeta^{(d_j-1)}$ ,  $1 \leq j \leq n$ , the  $d_j$ 's being the degrees of the basic homogeneous invariants of  $G$ .*

Theorem 3.8. was first obtained by Coxeter [7], who verified this fact for each group listed in Theorem 3.5. Coleman [8] supplied a general proof, using the fact that the number of reflections  $= \frac{1}{2} nh$ . This fact, which was at first known only by individual verification [7], was proven by Steinberg [20]. In view of Theorem 3.8, the numbers  $m_j = d_j - 1$  are usually referred to as the exponents of the group  $G$ .

We begin by proving Steinberg's result, needed for the proof of Coleman's theorem. We require a preliminary lemma and employ the following terminology. Let  $A = (a_{ij})$  be an  $n \times n$  matrix with non-negative entries. We associate with  $A$  a graph  $\mathcal{G}$  consisting of  $n$  nodes, connecting the nodes  $i, j$  iff  $a_{ij} > 0$ .  $A$  is said to be connected iff  $\mathcal{G}$  is connected.

**LEMMA 3.2.** Let  $A = (a_{ij})$  be a symmetric connected matrix. The largest eigenvalue  $\lambda$  of  $A$  is positive and a corresponding eigenvector  $e$  can be chosen all of whose entries are positive.

**REMARK.** The above is a special case of a theorem of Frobenius concerning the eigenvalues of matrices with non-negative entries [13]. Indeed the symmetry of  $A$  is not required. This extraneous assumption permits for a somewhat simpler proof and suffices for our purposes.

*Proof.* Let  $Q(x) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$  be the quadratic form associated with  $(a_{ij})$ . Then  $\lambda = \max_{\|x\|=1} Q(x) > 0$ , where  $\|x\|^2 = \sum_{i=1}^n x_i^2$ . Choose  $v = (v_1, \dots, v_n)$ ,  $\|v\| = 1$ , so that  $Q(v) = \lambda$  and let  $e = (e_1, \dots, e_n)$ , where  $e_i = |v_i|$ ,  $1 \leq i \leq n$ . Then  $e_i \geq 0$ ,  $1 \leq i \leq n$ , and  $\|e\| = 1$ . As all  $a_{ij} \geq 0$  and  $\|e\| = 1$ , we have  $\lambda = Q(v) \leq Q(e) \leq \lambda$ , so that  $Q(e) = \lambda$ . The latter implies  $Ae = \lambda e$ . It remains to show that each  $e_i > 0$ . Choose  $e_j > 0$ . Because of the connectivity assumption, we may choose  $i_1, \dots, i_r = j$  so that  $a_{i_1 j_1}, a_{j_1 j_2}, \dots, a_{j_{r-1} j}$  are all  $> 0$ . The relation  $\lambda e_{j_{r-1}} = \sum_{k=1}^n a_{j_{r-1} k} e_k$  shows that  $e_{j_{r-1}} > 0$ . Repeating this reasoning  $r$  times, we conclude that each  $e_i > 0$ .

THEOREM 3.9 (Steinberg [20]). Let  $h = \text{order of } R_1 \dots R_n$ ,  $r = \text{number of reflections in } G$ . Then  $r = \frac{nh}{2}$ .

*Proof.* We may label the walls of the fundamental region  $F$  so that  $W_1 \dots W_s$  are mutually perpendicular, and  $W_{s+1}, \dots, W_n$  are mutually perpendicular (I.e. if the nodes corresponding to  $W_1, \dots, W_s$  are black and those corresponding to  $W_{s+1}, \dots, W_n$  are white, then each black node is linked only to white nodes and conversely). Let  $E_1 = W_{s+1} \cap \dots \cap W_n$ ,  $E_2 = W_1 \cap \dots \cap W_s$ . Thus in terms of the dual basis  $\{r'_i\}$ ,  $E_1$  is the linear span of  $r'_1, \dots, r'_s$  and  $E_2$  the linear span of  $r'_{s+1}, \dots, r'_n$ . Let  $S = R_{s+1} \dots R_n$ ,  $T = R_1 \dots R_s$  and denote the orthogonal complement of  $E_i$ ,  $i = 1, 2$ , by  $E_i^\perp$ . The restriction of  $S$  to  $E_1$ , denoted by  $S_{E_1}$ , is the identity  $r_{s+1}, \dots, r_n$  form a basis for  $E_1^\perp$ . Since they are orthogonal to each other,  $R_i r_j = 0$  for  $i \neq j$ ,  $s+1 \leq i, j \leq n$ , so that  $S_{E_1}^\perp = -\text{identity}$ . Similarly  $T_{E_2} = \text{identity}$ ,  $T_{E_2}^\perp = -\text{identity}$ . We require the following

LEMMA 3.3. Let  $G_0$  be the  $n \times n$  matrix  $((r_i, r_j))$  and  $I$  the  $n \times n$  identity matrix.  $I - G_0$  is connected. Thus, by Lemma 3.2,  $I - G_0$  has a biggest positive eigenvalue  $\lambda$  and a corresponding eigenvector  $e$  with positive entries. Let  $\sigma = \sum_{i=1}^s e_i r'_i$ ,  $\tau = \sum_{i=s+1}^n e_i r'_i$ . The plane  $\pi$ , determined by  $\sigma$  and  $\tau$ , has non-trivial intersection with  $E_1^\perp$  and  $E_2^\perp$ . It follows that  $S_\pi(T_\pi)$  is a reflection of  $\pi$  in the line through  $\sigma$  ( $\tau$ ).

*Proof.* The entries of  $I - G_0$  are  $\geq 0$ , as  $(r_i, r_j) \leq 0$  whenever  $i \neq j$ . The irreducibility of  $G$  is equivalent to saying that  $I - G_0$  is connected. Let

$$G_0 = \begin{pmatrix} I & A \\ A' & I \end{pmatrix}, \quad G_0^{-1} = \begin{pmatrix} B & C \\ C' & D \end{pmatrix},$$

where  $A, C$  are  $s \times n - s$  matrices (we use  $I$  to denote the identity matrix for various degrees; here degree  $I = s$ ). The relations  $r_i = \sum_{j=1}^n (r_i, r_j) r'_j$ ,  $r'_i = \sum_{j=1}^n (r'_i, r'_j) r_j$ ,  $1 \leq i \leq n$ , show that  $G_0^{-1} = ((r'_i, r'_j))$ . Since  $G_0^{-1} G_0 = I$ , we have

$$(3.1) \quad BA + C = C' + DA' = 0$$

Let  $e^1$  be the vector consisting of the first  $s$  components of  $e$ ,  $e^2$  the vector

<sup>1)</sup> Geometrically, the directions of  $\sigma$ ,  $\tau$  are those in  $E_1$ ,  $E_2$  which produce the smallest angle. To prove this, one solves this minimum problem by the method of multipliers. Lagrange's equations lead to (3.2.).

consisting of the last  $n - s$  components of  $e$ . The equation  $(I - G_0) e = \lambda e$  becomes

$$(3.2) \quad A e^2 + \lambda e^1 = A' e^1 + \lambda e^2 = 0.$$

(3.1), (3.2) imply

$$(3.3) \quad \lambda B e^1 - C e^2 = \lambda D e^2 - C' e^1 = 0.$$

Let  $\sigma = \sum_{i=1}^s e_i r'_i$ ,  $\tau = \sum_{i=s+1}^n e_i r'_i$ . (3.3) may be rewritten as

$$(3.4) \quad \begin{aligned} r'_i \cdot (\lambda \sigma - \tau) &= 0, \quad 1 \leq i \leq s, \\ r'_i \cdot (\lambda \tau - \sigma) &= 0, \quad s + 1 \leq i \leq n. \end{aligned}$$

The vectors  $\lambda \sigma - \tau$ ,  $\lambda \tau - \sigma$  are  $\neq 0$  and in  $\pi$ . (3.4) states that  $\lambda \sigma - \tau \in E_1^\perp$ ,  $\lambda \tau - \sigma \in E_2^\perp$ . Since  $\sigma \in E_1$ ,  $\sigma' = \lambda \sigma - \tau \in E_1^\perp$ , we have  $S(\sigma) = \sigma$ ,  $S(\sigma') = -\sigma'$ . I.e.  $S_\pi$  is a reflection in the line through  $\sigma$ . Similarly,  $T_\pi$  is a reflection in the line through  $\tau$ .

We now return to the proof of Theorem 3.9. Let  $H$  be the subgroup generated by  $S, T$ .  $H_\pi$  is the group generated by  $S_\pi, T_\pi$ . Let

$$F_0 = \{v \mid v = x\sigma + y\tau, x, y > 0\} = F \cap \pi.$$

$F_0$  is a fundamental region for  $H_\pi$ . For let  $\gamma \in H$ ,  $\gamma_\pi \neq I$ . Then  $\gamma \neq I$  and we have  $\gamma_\pi F \cap F = \gamma F \cap F \cap \pi = \Phi$ .  $R_\pi$  is a rotation of  $\pi$  through twice the angle between  $\sigma$  and  $\tau$ . We show that  $\text{ord } R_\pi = h$ . For let  $\text{ord } R_\pi = k$ . Since  $R^h = I$ ,  $R_\pi^h = I$ , we have  $k \leq h$ . Choose  $p \in F_0$ .  $R^k(p) = R_\pi^k(p) = p$  so that  $R^k F \cap F \neq \Phi \Rightarrow R^k = I \Rightarrow h \leq k$ . Thus

$h = k$ . It follows that  $F_0$  is an angular wedge of angular width  $\frac{2\pi}{h}$  and

$H_\pi$  is a dihedral group of order  $2h$ . The  $h$  transforms of  $\sigma$  are contained in precisely  $(n-s)$  r.h.'s. The  $h$  transforms of  $\tau$  are contained in precisely  $s$  r.h.'s. Every r.h. of  $G$  has a non-trivial intersection with  $\pi$ . Since each of the transforms of  $F_0$  is contained in a chamber of  $G$  and each chamber is free of r.h.'s, these r.h.'s meet  $\pi$  only at the transforms of  $\sigma$  and  $\tau$ . Counting the r.h.'s at the transforms of  $\sigma$  and  $\tau$ , we obtain the count  $hs + h(n-s) = hn$ . Each r.h. is however counted twice, as it intersects  $\pi$  in a line and

thus meets two of the  $\sigma$  and  $\tau$  transforms. Hence  $r = \frac{hn}{2}$ .

As a by product of the above proof, we obtain the following result required to establish Theorem 3.8.



THEOREM 3.10.  $\zeta = e^{2\pi i/h}$  is an eigenvalue of  $R$ . Corresponding to  $\zeta$ , we may choose an eigenvector  $v$  not lying in any r.h. (Note: if  $v$  is complex, then  $v$  is said to lie in the r.h.  $\pi$  iff  $L(v) = 0$ ,  $L(x) = 0$  being the equation of  $\pi$ ).

*Proof.* Assume first that the  $R_i$ 's are labeled as in the proof of Theorem 3.9; i.e. the walls  $W_1, \dots, W_s$  are mutually perpendicular as are also  $W_{s+1}, \dots, W_n$ . Let  $\pi$  be the plane of Lemma 3.3. We choose two orthonormal vectors  $v_1, v_2$  in  $\pi$  such that  $v_1$  is not contained in any r.h. of  $G$  and

$$(3.5) \quad \begin{aligned} R(v_1) &= \cos \frac{2\pi}{h} v_1 + \sin \frac{2\pi}{h} v_2 \\ R(v_2) &= -\sin \frac{2\pi}{h} v_1 + \cos \frac{2\pi}{h} v_2 \end{aligned}$$

Let  $v = v_1 - iv_2$ . We conclude from (3.5) that  $R(v) = e^{2i\pi/h} v$ . Thus  $v$  is an eigenvector corresponding to the eigenvalue  $\zeta = e^{2i\pi/h}$ .  $v$  is not in any r.h. of  $G$  as  $v_1$  is not in any r.h. of  $G$ .

For an arbitrary labeling of indices, choose a permutation  $i_1, \dots, i_n$  of  $1, \dots, n$  so that the above reasoning applies to  $R' = R_{i_1} \dots R_{i_n}$ . By Theorem 3.7.  $R = R_1 \dots R_n = \sigma R' \sigma^{-1}$  for some  $\sigma \in G$ . Hence  $R(\sigma v) = \zeta(\sigma v)$ . Since the r.h.'s are permuted by  $\sigma$ , we conclude that  $\sigma v$  is also not contained in any r.h. of  $G$ .

We also require

THEOREM 3.11. 1 is not an eigenvalue of  $R$ .

REMARK. In Theorem 3.12 we obtain the characteristic equation of  $R$ , from which we may obtain Theorem 3.11. The following proof is shorter and avoids any explicit matrix representation for  $R$ .

*Proof.* Let  $\pi$  be the r.h. corresponding to the root  $r$  and  $\sigma$  the reflection in  $\pi$ . Then  $v' = \sigma v$  becomes

$$(3.6) \quad v' = v - 2(v, r)r$$

Suppose that  $R_1 \dots R_n v = v$ ,  $\Leftrightarrow R_2 \dots R_n v = R_1 v$ . Repeated application of (3.6) shows that  $R_2 \dots R_n v = v + \lambda_2 r_2 + \dots + \lambda_n r_n$ ,  $\lambda_2, \dots, \lambda_n$  being real numbers depending on  $v$ . Hence

$$(3.7) \quad v + \lambda_2 r_2 + \dots + \lambda_n r_n = v - 2(v, r_1)r_1$$

Since  $r_1, \dots, r_n$  are linearly independent we must have  $(v, r_1) = 0 \Leftrightarrow R_1 v = v$ , so that  $R_2 \dots R_n v = v$ . Repeating the reasoning, we con-



clude  $(v, r_i) = 0, 1 \leq i \leq n, \Rightarrow v = 0$ . Thus 1 is not an eigenvalue of  $R_1 \dots R_n$ .

We can now provide the

*Proof of Theorem 3.8.* Let  $v_1, \dots, v_n$  be linearly independent eigenvectors of  $R$  with  $v_1$  chosen as in Theorem 3.10; i.e.  $v_1$  corresponds to the eigenvalue  $\zeta = e^{2i\pi/h}$  and does not lie in any r.h. of  $G$ . Let  $x_1, \dots, x_n$  be a coordinate system adapted to  $v_1, \dots, v_n$ . As  $R^h = I$ , all eigenvalues of  $R$  are  $h$ -th roots of  $I$ . By Theorem 3.11, 1 is not an eigenvalue of  $R$ . Hence the eigenvalues of  $R$  are  $\zeta^{m_1}, \dots, \zeta^{m_n}$  where  $m_1 = 1$  and  $1 \leq m_1 \leq \dots \leq m_n = h - 1, 1 \leq i \leq n$ .  $R$  is given by  $x'_i = \zeta^{m_i} x_i, 1 \leq i \leq n$ .

Let  $I_1, \dots, I_n$  be a basic set of homogeneous invariants of  $G$  of respective degrees  $d_1 \leq \dots \leq d_n$ . By Theorem 2.5,

$$J = \frac{\partial(I_1, \dots, I_n)}{\partial(x_1, \dots, x_n)} \neq 0$$

off the r.h.'s of  $G$ . Hence  $J \neq 0$  whenever  $x = (x_1, 0, \dots, 0), x_1 \neq 0$ . It follows that there exists a permutation  $j = j(i)$  of 1 to  $n$  such that

$$\frac{\partial I_i}{\partial x_j}(x_1, 0, \dots, 0) \neq 0$$

for  $x_1 \neq 0$  and  $1 \leq i \leq n$ . This means that the  $x_1^{d_i-1}$  coefficient of

$$\frac{\partial I_i}{\partial x_j} \neq 0 \Rightarrow x_1^{d_i-1} x_j$$

coefficient of  $I_i \neq 0, 1 \leq i \leq n$ . Hence each  $x_1^{d_i-1} x_j$  is invariant under  $R$ . I.e.

$$(3.8) \quad (d_i - 1) + m_j \equiv 0 \pmod{h}, 1 \leq i \leq n$$

Rewrite (3.8) as

$$(3.9) \quad d_i - 1 = (h - m_j) + \varepsilon_i h, 1 \leq i \leq n$$

where each  $\varepsilon_i$  is an integer  $\geq 0$ . Let  $m'_j = h - m_j$ . The eigenvalues of  $R$  occur in pairs, so that the set of numbers  $\{m'_j\}$  is identical with  $\{m_j\}$ . Summing both sides of (3.9) from  $i = 1$  to  $i = n$ , we get

$$(3.10) \quad \sum_{i=1}^n (d_i - 1) = \sum_{j=1}^n m'_j + \left( \sum_{i=1}^n \varepsilon_i \right) h$$

By Theorem 2.2,  $\sum_{i=1}^n (d_i - 1) = r$ . Since

$$(3.11) \quad \sum_{j=1}^n m_j' = \sum_{j=1}^n (h - m_j) = nh - \sum_{j=1}^n m_j',$$

we also have  $\sum_{j=1}^n m_j' = \frac{nh}{2}$ . We conclude from Theorem 3.9 that

$$\sum_{i=1}^n (d_i - 1) = \sum_{j=1}^n m_j'. \quad (3.10) \text{ shows that } \sum_{i=1}^n \varepsilon_i = 0 \Rightarrow \varepsilon_i = 0, 1 \leq i \leq n.$$

It follows from (3.9) that  $d_i - 1 = m_i, 1 \leq i \leq n$ .

To make effective use of Coleman's Theorem, we need the explicit expression for the characteristic equation of  $R$ .

**THEOREM 3.12** (Coxeter [5], p. 218). *The characteristic equation of  $R = R_1 \dots R_n$  is given by*

$$(3.12) \quad \begin{vmatrix} \frac{1+\lambda}{2} & \lambda a_{12} & \dots & \lambda a_{1n} \\ a_{21} & \frac{1+\lambda}{2} & \lambda a_{23} & \dots & \lambda a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{n,n-1} & \frac{1+\lambda}{2} \end{vmatrix} = 0$$

where  $a_{ij} = -\cos(\pi/p_{ij}), 1 \leq i, j \leq n$ .

*Proof.* Let  $v = \sigma v'$  where  $\sigma$  is a reflection in the r.h. perpendicular to the root  $r$ .

Then

$$(3.13) \quad v = v' - 2(v' \cdot r) r$$

We use (3.13) to obtain the matrix for  $R_j$  relative to the basis  $r_1', \dots, r_n'$ .

Let  $v = \sum_{i=1}^n x_i r_i', v' = \sum_{i=1}^n x_i' r_i'$ . Then  $v' \cdot r_j = x_j', r_j = \sum_{i=1}^n a_{ij} r_i'$ .

Substituting into (3.13), we get

$$(3.14) \quad v = R_j v' \Leftrightarrow x_i = x_i' - 2a_{ij} x_j', 1 \leq i \leq n$$

Let

$$v = R_1 v^{(1)}, v^{(1)} = R_2 v^{(2)}, \dots, v^{(n-1)} = R_n v^{(n)}$$

so that  $v = R_1 \dots R_n v^{(n)}$ . Suppose that  $v^{(j)} = \sum_{i=1}^n x_i^{(j)} r'_i, 1 \leq j \leq n$ .

We conclude from (3.14) that

$$(3.15) \quad \left\{ \begin{array}{l} x_i = x_i' - 2a_{i1} x_1' \\ x_i' = x_i'' - 2a_{i2} x_2'' \\ \dots\dots\dots, 1 \leq i \leq n \\ x_i^{(n-1)} = x_i^{(n)} - 2a_{in} x_n^{(n)} \end{array} \right.$$

Let  $y_i = x_i^{(k)}, 1 \leq i \leq n$ . For each  $i$  we rewrite (3.15) as

$$(3.16) \quad \left\{ \begin{array}{l} x_i' - x_i = 2a_{i1} y_1 \\ x_i'' - x_i' = 2a_{i2} y_2 \\ \dots\dots\dots \\ y_i - x_i^{(i-1)} = 2a_{ii} y_i \end{array} \right. \quad (3.17) \quad \left\{ \begin{array}{l} x_i^{(i+1)} - y_i = 2a_{i,i+1} y_{i+1} \\ x_i^{(i+2)} - x_i^{(i+1)} = 2a_{i,i+2} y_{i+2} \\ \dots\dots\dots \\ x_i^{(n)} - x_i^{(n-1)} = 2a_{in} y_n \end{array} \right.$$

Adding up respectively the equations in (3.16), and (3.17), we obtain

$$(3.18) \quad -x_i = \sum_{j=1}^{i-1} 2a_{ij} y_j + y_i, 1 \leq i \leq n$$

$$(3.19) \quad x_i^{(n)} = \sum_{j=i+1}^n 2a_{ij} y_j + y_i, 1 \leq i \leq n$$

(3.18), (3.19) may be abbreviated as

$$(3.20) \quad -x = Ay, x^{(n)} = A' y$$

where

$$(3.21) \quad A = \begin{bmatrix} 1 & & & & \\ 2a_{21} & & & & \\ . & 1 & & & \\ . & . & . & & \\ . & . & . & . & \\ 2a_{n1} & . & . & 2a_{n, n-1} & 1 \end{bmatrix}$$

the entries above the diagonal being zero.

Hence  $x = -A(A')^{-1} x^{(n)}$ , so that  $-A(A')^{-1}$  is the matrix for  $R = R_1 \dots R_n$  relative to the basis  $r'_1, \dots, r'_n$ . The characteristic equation for  $R$  is thus given by

$$(3.22) \quad | -A(A')^{-1} - \lambda I | = 0 \Leftrightarrow \left| \frac{A + \lambda A'}{2} \right| = 0$$

which is the same as (3.12).

We rewrite the characteristic equation in a more symmetric form. Suppose first that  $G$  is of type  $I$ . We label nodes of the graphs in diagram 3.2 from left to right as  $1, \dots, n$ . Thus  $a_{ij} = 0$  whenever  $|j - i| > 1$ . Multiplying first the  $i$ -th row of the determinant in (3.12) by  $\lambda^{(i-1)/2}$ ,  $1 \leq i \leq n$ , then the  $j$ -th column by  $\lambda^{-j/2}$ ,  $1 \leq j \leq n$ , we get

$$(3.23) \quad \begin{vmatrix} A & & & \\ & \cdot & & a_{ij} \\ & & \cdot & \\ & & & \cdot \\ a_{ij} & & & \\ & & & A \end{vmatrix} = 0$$

where  $A = \frac{\lambda^{1/2} + \lambda^{-1/2}}{2}$

If  $G$  is of type  $II$ , then the nodes on the principal chain are labeled from left to right as  $1$  to  $n - 1$ , the remaining node being labeled  $n$ . The  $n^{\text{th}}$  node is linked to the  $q^{\text{th}}$  node. Let  $i' = i, j' = j$ ,  $1 \leq i, j \leq n - 1$ , and  $i' = j' = q + 1$  whenever  $i$  or  $j = n$ . Multiply first the  $i$ -th row of the determinant in (3.12) by  $\lambda^{\frac{i'-1}{2}}$ ,  $1 \leq i \leq n$ , then the  $j$ -th column by  $\lambda^{-j'/2}$ . We obtain again (3.23). We have proven

**COROLLARY.** *The characteristic equation of  $R$  is given by (3.23).*

We illustrate the use of Coleman's Theorem by computing the  $d_i$ 's for the icosahedral group  $I_3$ . In this case the characteristic equation (3.23) becomes

$$(3.24) \quad \begin{vmatrix} A & -\frac{1}{2} & 0 \\ -\frac{1}{2} & A & -\cos \frac{\pi}{5} \\ 0 & -\cos \frac{\pi}{5} & A \end{vmatrix} = 0$$

The roots of (3.24) are readily computed to be  $\zeta = e^{\frac{2\pi i}{10}}, \zeta^5, \zeta^9$ . It follows from Coleman's Theorem that  $d_1 = 2, d_2 = 6, d_3 = 10$ .

### 3. TABULATION OF THE DEGREES

Theorem 3.8 can be used to compute the degrees of the basic homogeneous invariants of  $G$ , in case  $G$  is an irreducible reflection group acting on  $R^n$ . This has been done in [7], and we tabulate these degrees below

Group	$d_1, \dots, d_n$
$A_n$ ( $n \geq 1$ )	$2, \dots, n + 1$
$B_n$ ( $n \geq 2$ )	$2, 4, \dots, 2n$
$D_n$ ( $n \geq 4$ )	$2, 4, \dots, n, \dots, 2n - 4, 2n - 2$
$H_2^n$ ( $n \geq 5$ )	$2, n$
$E_6$	$2, 5, 6, 8, 9, 12$
$E_7$	$2, 6, 8, 10, 12, 14, 18$
$E_8$	$2, 8, 12, 14, 18, 20, 24, 30$
$F_4$	$2, 6, 8, 12$
$I_3$	$2, 6, 10$
$I_4$	$2, 12, 20, 30$

We observe that in each case,  $d_1 = 2$ . This can be seen as follows. Suppose that there existed a homogeneous invariant  $I(x)$  of degree 1. Since  $I(\sigma x) = I(x)$  whenever  $\sigma \in G$ , the hyperplane  $\{x \mid I(x) = 0\}$  would be a proper invariant subspace of  $G$ , contradicting that the latter is irreducible. Hence there are no homogeneous invariants of degree 1 and  $d_1 \geq 2$ . On the other hand,  $\sum_{i=1}^n x_i^2$  is invariant under  $G$  as  $G$  is orthogonal. It follows

that  $d_1 = 2$ , with corresponding invariant  $I_1 = \sum_{i=1}^n x_i^2$ .

In applying Theorem 3.8, we must find the roots of the characteristic equation (3.23). In some cases, this is a rather tedious computation. For the groups  $A_n, B_n, D_n, H_2^n$  we can exhibit a basis of homogeneous invariants without the use of Theorem 3.8. We require

**THEOREM 3.13.** *Let  $G$  be a finite reflection group acting on the  $n$ -dimensional vector space  $V$  over a given field  $k$ . Let  $P_1, \dots, P_n$  be homogeneous*

invariants of  $G$  of respective degrees  $k_1, \dots, k_n$ .  $P_1, \dots, P_n$  form a basis for the invariants of  $G \Leftrightarrow k_1 \dots k_n = |G|$  and

$$\Delta = \frac{\partial (P_1, \dots, P_n)}{\partial (x_1, \dots, x_n)} \neq 0.$$

*Proof.* By relabeling indices, we may assume  $k_1 \leq \dots \leq k_n$ . The  $\Rightarrow$  part of the theorem is contained in Theorems 1.2, 2.2, 2.3. Conversely, let  $k_1 \dots k_n = |G|$  and  $\Delta \neq 0$ . Thus  $P_1, \dots, P_n$  are algebraically independent. Let  $I_1, \dots, I_n$  be basic homogeneous invariants of respective degrees  $d_1, \dots, d_n$ . Suppose  $k_i = d_i$ ,  $1 \leq i \leq i_0$ , but  $k_{i_0+1} < d_{i_0+1}$ . Then  $P_1, \dots, P_{i_0+1}$  are polynomials in  $I_1, \dots, I_{i_0}$ , implying that  $P_1, \dots, P_n$  are algebraically dependent, a contradiction. Hence  $k_i \geq d_i$ ,  $1 \leq i \leq n$ . Since

$$\prod_{i=1}^n d_i = \prod_{i=1}^n k_i = |G|, \text{ we must have } k_i = d_i, 1 \leq i \leq n.$$

Let  $\delta_m = \dim \mathcal{J}_m$ ,  $0 \leq m < \infty$ ,  $\mathcal{J}_m$  being the space of homogeneous invariants of degree  $m$ . Then  $\delta_m =$  number of non-negative integral solutions to  $j_1 d_1 + \dots + j_n d_n = m$ . This number also equals the number of monomials  $P_1^{j_1} \dots P_n^{j_n}$  which are of degree  $m$ . The algebraic independence of  $P_1, \dots, P_n$  implies that these  $\delta_m$  monomials are linearly independent over  $k$ . Thus  $\mathcal{J}_m$  is spanned by these monomials for  $0 \leq m < \infty$ . We have shown that every homogeneous invariant is a polynomial in  $P_1, \dots, P_n$ , so that the  $P_i$ 's form a basis for the invariants of  $G$ .

We now obtain an explicit basis for the invariants of  $A_n, B_n, D_n, H_2^n$ .  $A_n$ : This group consists of the  $(n+1)!$  permutations  $x'_i = x_{\sigma(i)}$ ,  $1 \leq i \leq n+1$ , restricted to the subspace  $V = \{x \mid x_1 + \dots + x_{n+1} = 0\}$ . We choose  $x_1, \dots, x_n$  as coordinates on  $V$ . Let  $P_i = \sum_{j=1}^{n+1} x_j^{i+1}$ ,  $1 \leq i \leq n$ , where  $x_{n+1} = -(x_1 + \dots + x_n)$ .  $P_i$  is a homogeneous invariant of degree  $i+1$ . We have  $2 \cdot \dots \cdot (n+1) = (n+1)! = |A_n|$ .

We show that  $\Delta \neq 0$ . Now

$$\frac{\partial P_i}{\partial x_j} = (i+1) x_j^i - (i+1) x_{n+1}^i, 1 \leq i, j \leq n.$$

Hence  $\Delta = (n+1)! D$  where  $D$  is the  $n \times n$  determinant whose  $(ij)$ -th entry  $= x_j^i - x_{n+1}^i$ . To evaluate  $D$ , we introduce the Vandermonde determinant

$$\begin{vmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_{n+1} \\ x_1^n & \dots & x_{n+1}^n \end{vmatrix} = \prod_{1 \leq i < j \leq n+1} (x_j - x_i)$$

Subtracting the  $(n+1)$ -th column from the first  $n$  columns, the above determinant is readily seen to equal  $(-1)^n D$ . Thus

$$(3.25) \quad \Delta = (-1)^{n+2} (n+1)! \prod_{1 \leq i < j \leq n+1} (x_j - x_i) = \\ (n+1)! \prod_{1 \leq j \leq n} (x_j - x_i) \cdot \prod_{i=1}^n (x_i + s)$$

where  $s = x_1 + \dots + x_n$ . (3.25) shows that  $\Delta \neq 0$ . We conclude that  $d_1 = 2, \dots, d_n = n+1$ .

$B_n$ : Let  $P_i = \sum_{j=1}^n x_j^{2i}$ ,  $1 \leq i \leq n$ .  $P_i$  is a homogeneous invariant of degree  $2i$ . We have  $2 \cdot \dots \cdot 2n = 2^n n! = |B_n|$ . A computation shows that

$$\Delta = 2^n n! \prod_{i=1}^n x_i \prod_{1 \leq i < j \leq n} (x_j^2 - x_i^2) \neq 0. \text{ It follows that } d_1 = 2, \dots, d_n = 2n.$$

$D_n$ : Let  $P_1 = x_1 \dots x_n$ ,  $P_i = \sum_{j=1}^n x_j^{2(i-1)}$ ,  $2 \leq i \leq n$ .  $P_1$  is a homogeneous invariant of degree  $n$ ;  $P_i$ ,  $2 \leq i \leq n$ , is a homogeneous invariant of degree  $2(i-1)$ . The product of the degrees  $= n \cdot 2 \cdot 4 \cdot \dots \cdot (2n-2) = 2^{n-1} n! = |D_n|$ .

$$(3.26) \quad \Delta = \begin{vmatrix} \frac{P_1}{x_1} & \dots & \frac{P_1}{x_n} \\ 2x_1 & \dots & 2x_n \\ \cdot & \dots & \cdot \\ 2(n-1)x_1^{2n-3} & \dots & 2(n-1)x_n^{2n-3} \end{vmatrix} \\ = 2^{n-1} (n-1)! \prod_{1 \leq i < j \leq n} (x_j^2 - x_i^2) \neq 0$$

It follows that  $d_i, \dots, d_n$  are identical with the numbers  $2, 4, \dots, n, \dots, 2n-4, 2n-2$ .

$H_2^n$ : Let  $z$  be the complex coordinate  $x_1 + i x_2$ .  $H_2^n$  may be described as the group generated by the transformation  $z \rightarrow \bar{z}$ ,  $z \rightarrow \zeta z$ , where  $\zeta = e^{\frac{2\pi i}{n}}$ . Let  $P_1 = x_1^2 + x_2^2$ ,  $P_2 = \operatorname{Re} z^n$ .  $P_1, P_2$  are homogeneous invariants of respective degrees  $2, n$ . The product of these degrees  $= 2n = |H_2^n|$ . A computation yields

$$\frac{\partial (P_1, P_2)}{\partial (x_1, x_2)} = -2n \operatorname{Im} z^n \neq 0.$$

It follows that  $d_1 = 2, d_2 = n$ .

#### 4. SOLOMON'S THEOREM

We present in this section another method for determining the degrees of the basic invariants, valid whenever the underlying field  $k$  has characteristic 0.

**THEOREM 3.14** (Solomon [18]). *Let  $G$  be a finite reflection group acting on the  $n$ -dimensional space  $V$ . Let  $g_r$  = number of elements of  $G$  which fix some  $r$ -dimensional subspace of  $V$  but do not fix a subspace of higher dimension. Let  $d_1, \dots, d_n$  be the degrees of the basic homogeneous invariants of  $G$  and set  $m_j = d_j - 1$ . Then*

$$(3.27) \quad (t + m_1) \dots (t + m_n) = g_0 + g_1 t + \dots + g_n t^n$$

Equating the  $t^{n-1}$ -coefficients of both sides of (3.27), we obtain  $g_1 = r = \sum_{i=1}^n m_i$ . Setting  $t = 1$  in (3.27), we obtain  $\prod_{i=1}^n (m_i + 1) = \sum_{i=0}^n g_i = |G|$ . Thus Theorem 3.14 generalizes Theorem 2.2.

To prove Theorem 3.14, we obtain an analog of Molien's formula for the invariant differential forms of  $G$ . We digress to a brief discussion of differential forms.

For  $p > 0$ , let  $\omega = \sum_{i_1 < \dots < i_p} r_{i_1 \dots i_p}(x) dx_{i_1} \dots dx_{i_p}$ , where  $r_{i_1 \dots i_p}(x) \in k(x)$ , the summation extending over all integer  $p$ -tuples satisfying  $1 \leq i_1 < \dots < i_p \leq n$ .  $\omega$  is called a differential  $p$ -form (or simply  $p$ -form). The elements of  $k(x)$  are called the 0-forms. If  $\eta = \sum_{i_1 < \dots < i_p} s_{i_1 \dots i_p}(x) dx_{i_1} \dots dx_{i_p}$  is another  $p$ -form, then we define

$$\omega + \eta = \sum_{i_1 < \dots < i_p} (r_{i_1 \dots i_p} + s_{i_1 \dots i_p}) dx_{i_1} \dots dx_{i_p}.$$

Thus the  $p$ -forms constitute a vector space over  $k(x)$  which we denote by  $\mathcal{D}_p$ . The elements  $dx_{i_1} \dots dx_{i_p}$  form a basis for  $\mathcal{D}_p$ , so that  $\dim \mathcal{D}_p = \binom{n}{p}$ ,  $0 \leq p \leq n$ . We also define a multiplication between two forms as follows. Let  $dx_i dx_j = -dx_j dx_i$ ; in particular  $dx_i dx_i = 0$ . The product  $\omega\eta$  of any two forms  $\omega, \eta$  is then obtained by the distributive law. We observe that for 1-forms,  $\omega\eta = -\eta\omega$ , so that  $\omega\omega = 0$ . It follows that  $\mathcal{D}_p = 0$  for  $p > n$ . Finally, for any rational function  $r$ , we define the 1-form  $dr$  to be

$$\sum_{i=1}^n \frac{\partial r}{\partial x_i} dx_i.$$



It is then readily checked that for  $n$  rational functions,  $r_1, \dots, r_n$ , we have

$$dr_1 \dots dr_n = \frac{\partial (r_1, \dots, r_n)}{\partial (x_1, \dots, x_n)} dx_1 \dots dx_n.$$

Let  $\sigma$  be a non-singular matrix with entries in  $k$ . We define

$$\sigma \omega = \sum_{i_1 < \dots < i_p} r_{i_1} \dots r_{i_p} (\sigma^{-1}x) dx_{i_1} (\sigma^{-1}x) \dots dx_{i_p} (\sigma^{-1}x)$$

Thus  $\sigma$  becomes a linear transformation on each  $\mathcal{D}_p$ , interpreting the latter as a vector space over  $k$ . Let  $k^n$  be the space of  $n$ -tuples with entries in  $k$ . If  $G$  is a group of linear transformations acting on  $k^n$ , then  $\omega$  is said to be invariant under  $G$  provided  $\sigma \omega = \omega$ ,  $\forall \sigma \in G$ .

We shall prove Theorem 3.14 describing the invariant differential forms with polynomial coefficients.  $G$  is assumed throughout to be a finite reflection group acting on  $k^n$ .

LEMMA 3.4. Let  $I_1, \dots, I_n$  be basic homogeneous invariants for  $G$ . Let

$$\Pi(x) = \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)}.$$

The polynomial  $p(x)$  satisfies  $\sigma p = (\det \sigma) p$ , for every  $\sigma \in G$  (in which case, we say  $p$  is skew) iff  $p = \Pi i$  where  $i$  is a polynomial invariant under  $G$ .

*Proof.* Let  $y = \sigma x$ . Then

$$\begin{aligned} (3.28) \quad \Pi(x) &= \frac{\partial (I_1(y), \dots, I_n(y))}{\partial (x_1, \dots, x_n)} \\ &= \frac{\partial (I_1(y), \dots, I_n(y))}{\partial (y_1, \dots, y_n)} \det \sigma = \Pi(\sigma x) \det \sigma \end{aligned}$$

which shows that  $\Pi$  is skew. Hence  $\Pi i$  is skew for every invariant polynomial  $i$ .

Conversely, let  $p(x)$  be skew. Let  $\pi$  be an r.h. of  $G$  with equation  $L(x) = 0$ . By Lemma 2.2, we may choose  $v \notin \pi$ , so that  $v$  is a common eigenvector to all reflections in  $G$  with r.h.  $\pi$ . Choose  $x = Ty$ ,  $\det T \neq 0$ , so that in the  $y$  coordinates the equation of  $\pi$  becomes  $y_n = 0$  and  $v$  becomes  $(0, \dots, 0, 1)$ . Let  $q(y) = p(Ty)$ . Let  $H$  be the subgroup of  $G$  which fixes  $\pi$ . By Lemma 2.2,  $H$  is a cyclic group. Let  $\sigma$  generate  $H$  and  $h = \text{ord } H$ . If  $\zeta$  is the eigenvalue of  $\sigma$  which is a primitive  $h$ -th root of 1, then

$q(y_1, \dots, y_{n-1}, \zeta y_n) = \zeta^{-1} q(y_1, \dots, y_n)$ . Writing  $q = \sum q_i y_n^i$ , the  $q_i$ 's being polynomials in  $y_1, \dots, y_{n-1}$ , we obtain

$$(3.29) \quad \sum q_i \zeta^{i+1} y_n^i = \sum q_i y_n^i$$

Equating coefficients in (3.29), we conclude  $q_i = 0$  whenever  $h \nmid i+1$ . Thus  $q_i = 0$  for  $i < h-1 \Rightarrow y_n^{h-1} \mid q \Rightarrow L^{h-1} \mid p$ . Repeating this argument for all r.h.'s of  $G$  and using Theorem 2.5, we conclude that  $P = \Pi i$ , where  $i$  is a polynomial.  $\sigma i = \sigma P / \sigma \Pi = \frac{P}{\Pi} = i$  shows that  $i$  is invariant under  $G$ .

LEMMA 3.5. Let  $\sigma$  be a non-singular matrix with entries in  $k$ . Let  $r \in k(x)$ . Then  $\sigma(dr) = d(\sigma r)$ .

*Proof.* By definition

$$(3.30) \quad \sigma(dr) = \sum_{i=1}^n \frac{\partial r}{\partial x_i} (\sigma^{-1}x) dx_i (\sigma^{-1}x), \quad d(\sigma r) = \sum_{i=1}^n \frac{\partial}{\partial x_i} (r(\sigma^{-1}x)) dx_i$$

$$\text{Let } \sigma^{-1} = (a_{ij}). \text{ Then } x_i (\sigma^{-1}x) = \sum_{j=1}^n a_{ij} x_j \text{ and } \frac{\partial x_i}{\partial x_j} (\sigma^{-1}x) = a_{ij}.$$

Hence

$$(3.31) \quad dx_i (\sigma^{-1}x) = \sum_{j=1}^n a_{ij} dx_j$$

Applying the chain rule,

$$(3.32) \quad \frac{\partial}{\partial x_i} (r(\sigma^{-1}x)) = \sum_{j=1}^n \frac{\partial r}{\partial x_j} (\sigma^{-1}x) a_{ji}$$

Inserting (3.31), (3.32) into (3.30), we get  $\sigma(dr) = d(\sigma r)$ .

THEOREM 3.15. Every invariant  $p$ -form with polynomial coefficients may be expressed uniquely as

$$\sum_{i_1 < \dots < i_p} a_{i_1 \dots i_p} dI_{i_1} \dots dI_{i_p}, \quad a_{i_1 \dots i_p} \in k[I_1, \dots, I_n].$$

*Proof.* By Lemma 3.5,  $\sigma(dI_k) = dI_k$ , so that  $dI_1, \dots, dI_n$  are invariant forms. Since  $\sigma(\omega\eta) = \sigma(\omega)\sigma(\eta)$  for any two forms  $\omega, \eta$ , we conclude that

$$\sum_{i_1 < \dots < i_p} a_{i_1 \dots i_p} dI_{i_1} \dots dI_{i_p} \text{ is invariant whenever } a_{i_1 \dots i_p} \in k(I_1, \dots, I_n).$$

We show that the  $\binom{n}{p}$  forms  $dI_{i_1} \dots dI_{i_p}$  are linearly independent over  $k(x)$ , so that they form a basis for  $\mathcal{D}_p$  over  $k(x)$ . Suppose that

$$\sum_{i_1 < \dots < i_p} k_{i_1 \dots i_p} dI_{i_1} \dots dI_{i_p} = 0, \quad k_{i_1 \dots i_p} \in k(x).$$

Multiply this relation by  $dI_{i_{p+1}} \dots dI_{i_n}$ , where  $i_{p+1}, \dots, i_n$  are the indices complementary to  $i_1, \dots, i_p$ . We obtain

$$k_{i_1 \dots i_p} dI_1 \dots dI_n = k_{i_1 \dots i_p} \Pi(x) dx_1 \dots dx_n = 0 \Rightarrow k_{i_1 \dots i_p} = 0$$

for all  $i_1, \dots, i_p$ . Hence the  $\binom{n}{p}$  forms  $dI_{i_1} \dots dI_{i_p}$  are linearly independent over  $k(x)$ . It follows that every  $p$ -form  $\omega$  may be expressed uniquely as

$$\omega = \sum_{i_1 < \dots < i_p} a_{i_1 \dots i_p} dI_{i_1} \dots dI_{i_p}, \quad a_{i_1 \dots i_p} \in k(x).$$

If  $\omega$  is invariant, then the group averaging argument shows that  $a_{i_1 \dots i_p} \in k(I_1, \dots, I_n)$ . Multiply both sides of the above relation by  $dI_{i_{p+1}} \dots dI_{i_n}$ . We get

$$(3.33) \quad \omega dI_{i_{p+1}} \dots dI_{i_n} = \pm \Pi a_{i_1 \dots i_p} dx_1 \dots dx_n.$$

Let  $\omega$  be a  $p$ -form with polynomial coefficients. We conclude from (3.33) that  $\Pi a_{i_1 \dots i_p}$  is a polynomial. Since  $\Pi a_{i_1 \dots i_p}$  is skew, Lemma 3.4 implies that  $\Pi a_{i_1 \dots i_p} = \Pi i$ ,  $i$  being an invariant polynomial. Hence  $a_{i_1 \dots i_p} \in k[I_1, \dots, I_n]$  for all  $i_1, \dots, i_p$ , thus proving Theorem 3.11.

**THEOREM 3.16.** *Let  $\sigma_p(x_1, \dots, x_n)$  be the  $p$ -th elementary symmetric function in  $x_1, \dots, x_n$  ( $\sigma_0$  is interpreted to be 1). Let  $\omega_1(\gamma), \dots, \omega_n(\gamma)$  be the eigenvalues of  $\gamma$ ,  $\gamma \in G$ . Then*

$$(3.34) \quad \frac{\sigma_p(t^{m_1}, \dots, t^{m_n})}{(1-t^{m_1+1}) \dots (1-t^{m_n+1})} = \frac{1}{|G|} \sum_{\gamma \in G} \frac{\sigma_p(\omega_1(\gamma), \dots, \omega_n(\gamma))}{(1-\omega_1(\gamma)t) \dots (1-\omega_n(\gamma)t)}, \quad 0 \leq p \leq n$$

**REMARK.** For  $p = 0$ , the above becomes formula (2.5) of Chapter II.

*Proof.* Let  $\mathcal{D}_{pm}$  = space of  $p$ -forms whose coefficients are homogeneous polynomials of degree  $m$ .  $\mathcal{D}_{pm}$  is a finite dimensional vector space over  $k$ . Let  $\mathcal{J}_{pm}$  = space of invariant forms in  $\mathcal{D}_{pm}$  and  $d_{pm} = \dim \mathcal{J}_{pm}$ . For  $0 \leq p \leq n$ , let  $p_p(t) = \sum_{m=0}^{\infty} d_{pm} t^m$ . We obtain two formulas for  $p_p(t)$  by computing  $d_{pm}$  in two different ways. By Theorem 3.15, the differentials

$$I_1^{k_1} \dots I_n^{k_n} dI_{i_1} \dots dI_{i_p}, \quad m = k_1(m_1+1) \dots + k_n(m_n+1) + m_{i_1} + \dots + m_{i_p},$$

form a basis for  $\mathcal{J}_{pm}$ , so that

$$(3.35) \quad p_p(t) = \frac{\sigma_p(t^{m_1}, \dots, t^{m_n})}{(1-t^{m_1+1}) \dots (1-t^{m_n+1})}$$

Let  $\tilde{k}$  = algebraic closure of  $k$ . Define  $\tilde{\mathcal{D}}_{pm}, \tilde{\mathcal{J}}_{pm}$ , analogously to  $\mathcal{D}_{pm}, \mathcal{J}_{pm}$ , replacing  $k$  by  $\tilde{k}$ . For  $\gamma \in G$ ,  $\gamma$  acts both on  $\mathcal{D}_{pm}$  and  $\tilde{\mathcal{D}}_{pm}$ . Let  $(\text{Tr } \gamma)_{pm}$  = trace of  $\gamma$  as a transformation on  $\mathcal{D}_{pm}$  = trace of  $\gamma$  as a transformation on  $\tilde{\mathcal{D}}_{pm}$ . By Lemma 1.2

$$(3.36) \quad d_{pm} = \frac{1}{|G|} \sum_{\gamma \in G} (\text{Tr } \gamma)_{pm}$$

Choose  $T$  so that  $T \sigma T^{-1} = D$ ,  $D$  being diagonal with diagonal entries  $\omega_1(\gamma), \dots, \omega_n(\gamma)$ . The elements  $x^a dx_{i_1} \dots dx_{i_p}$ ,  $|a| = m$  and  $1 \leq i_1 < \dots < i_p \leq n$ , form a basis for  $\tilde{\mathcal{D}}_{pm}$ . Since

$$(3.37) \quad D(x^a dx_{i_1} \dots dx_{i_p}) = [\omega(\gamma^{-1})]^a \omega_{i_1}(\gamma^{-1}) \dots \omega_{i_p}(\gamma^{-1}),$$

we have

$$(3.38) \quad (\text{Tr } D)_{pm} = \sum_{|a|=m} [\omega(\gamma^{-1})]^m \sigma_p(\omega(\gamma^{-1}))$$

(3.36), (3.38) yield

$$(3.39) \quad d_{pm} = \frac{1}{|G|} \sum_{\gamma \in G} \sum_{|a|=m} [\omega(\gamma)]^a \sigma_p[\omega(\gamma)]$$

so that

$$(3.40) \quad p_p(t) = \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{r \in G} \sum_{|a|=m} [\omega(\gamma)]^a \sigma_p(\omega(\gamma)) t^m \\ = \frac{1}{|G|} \sum_{\gamma \in G} \frac{\sigma_p(\omega(\gamma))}{(1-\omega_1(\gamma)t) \dots (1-\omega_n(\gamma)t)}$$

(3.34) follows from (3.35) and (3.40).

We derive from (3.34) the following identity.

THEOREM 3.17. For  $1 \leq p \leq n$ ,

$$(3.41) \quad \sum_{i_1 < \dots < i_p} \frac{t^{mi_1 + \dots + mi_p}}{(1-t^{mi_1+1}) \dots (1-t^{mi_p+1})} \\ = \frac{1}{|G|} \sum_{\gamma \in G} \sum_{i_1 < \dots < i_p} \frac{\omega_{i_1}(\gamma) \dots \omega_{i_p}(\gamma)}{(1-\omega_{i_1}(\gamma)t) \dots (1-\omega_{i_p}(\gamma)t)}$$

*Proof.* One verifies readily, for  $1 \leq p \leq n$ , the identity

$$(3.42) \quad \sum_{i_1 < \dots < i_p} \frac{u_{i_1} \dots u_{i_p}}{(1 - u_{i_1} t) \dots (1 - u_{i_p} t)} \\ = \frac{h_{p1}(t) \sigma_1(u_1, \dots, u_n) + \dots + h_{pn}(t) \sigma_n(u_1, \dots, u_n)}{(1 - u_1 t) \dots (1 - u_n t)}$$

the  $u_i$ 's being indeterminates and the  $h_{pi}$ 's being polynomials in  $t$ . Substitute for  $u_i$ ,  $\omega_i(\gamma)$  and average over the group. By Theorem 3.16, the group average becomes expression (3.42),  $u_i$  being replaced by  $t^{m_i}$ , thus proving (3.41).

We can now provide the

*Proof of Theorem 3.14.* Expand both sides of (3.41) in powers of  $1 - t$  and equate the coefficients of  $(1 - t)^{-p}$ . For the left side this coefficient is

$$\sum_{i_1 < \dots < i_p} \frac{1}{(m_{i_1} + 1) \dots (m_{i_p} + 1)}$$

Let  $\gamma$  be an element which fixes an  $r$  dimensional subspace, but does not fix a higher dimensional subspace. This means that precisely  $r$  of the eigenvalues of  $\gamma$  equal 1.  $\gamma$  contributes to the coefficient of  $(1 - t)^{-p}$  on the right side of (3.41) iff  $r \geq p$ , the contribution being  $\binom{r}{p}$ . It follows that for the right side, the  $(1 - t)^{-p}$  coefficient is  $\frac{1}{|G|} \sum_{r=0}^n \binom{r}{p} g_r$ . Since  $\prod_{i=1}^n (m_i + 1) = |G|$ , we conclude that

$$(3.43) \quad \sum_{r=0}^n \binom{r}{p} g_r = \sum_{i_1 < \dots < i_{n-p}} (m_{i_1} + 1) \dots (m_{i_{n-p}} + 1), \quad 1 \leq p \leq n$$

Note that for  $p = 0$ , (3.43) becomes  $|G| = (m_1 + 1) \dots (m_n + 1)$ . Hence (3.43) also holds for  $p = 0$ .

The left and right side of (3.43) equal respectively  $\frac{1}{p!}$  ( $p$ -th derivative at  $t = 1$ ) of  $g_0 + \dots + g_n t^n$ ,  $(t + m_1) \dots (t + m_n)$ . Thus  $(t + m_1) \dots (t + m_n) = g_0 + \dots + g_n t^n$ .