

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 24 (1978)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INVARIANTS OF FINITE REFLECTION GROUPS
Autor: Flatto, Leopold
Kapitel: 4. Decomposition of Finite Reflection Groups
DOI: <https://doi.org/10.5169/seals-49704>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 27.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Let $J_i = \sum A_m y_n^m$, the A_m 's being polynomials in y_1, \dots, y_{n-1} . (2.20) implies that $A_m = 0$ whenever $h \nmid m$, so that $A_m = 0, 0 \leq m \leq h - 1$. Since

$$\frac{\partial J_i}{\partial y_m} = \sum_m A_m y_n^{m-1},$$

we conclude

$$y_n^{h-1} \left| \frac{\partial J_i}{\partial y_n}, 1 \leq i \leq n. \right.$$

Hence

$$(2.21) \quad y_n^{h-1} \left| \frac{\partial (J_1, \dots, J_n)}{\partial (y_1, \dots, y_n)}, \right.$$

Since

$$\frac{\partial (J_1, \dots, J_n)}{\partial (y_1, \dots, y_n)} = J(x) \cdot \det \tau,$$

(2.21) is equivalent to $L^{h-1}(x) \mid J(x)$. It follows that if $L_i(x) = 0, 1 \leq i \leq r$, are the r.h.'s for G , then $\prod_{i=1}^r L_i \mid J$. But $J, \prod_{i=1}^r L_i$ have the same degree r , so that $J = c \prod_{i=1}^r L_i, c \neq 0$.

4. DECOMPOSITION OF FINITE REFLECTION GROUPS

We shall decompose every finite reflection group into a direct product of irreducible ones and show that it suffices to study the invariant theory of the irreducible groups.

DEFINITION 2.3. Let the group G act on V . G is said to be reducible iff there exists a proper subspace W invariant under G ; i.e. $\sigma w \in W$ for $\sigma \in G, w \in W$. G is said to be completely reducible iff $V = V_1 \oplus V_2, V_1$ and V_2 being proper invariant subspaces. G is said to be irreducible iff it is not reducible.

THEOREM 2.6. (Maschke [22], Vol. 2, p. 179). *Let G be a finite group acting on the vector space V . If G is reducible, then it is completely reducible.*

Proof. Let V_1 be a proper invariant subspace of V . Let V_2 be a complementary subspace. Thus for $v \in V$, we have a unique decomposition

$v = v_1 + v_2, v_i \in V_i (i=1, 2)$. Let $\eta v = v_2$ and set $\tau = \frac{1}{|G|} \sum_{\sigma \in G} \sigma \eta \sigma^{-1}$.

τ satisfies the following:

- i) $\tau \sigma = \sigma \tau, \sigma \in G$. For $\sigma \tau = \frac{1}{|G|} \sum_{\sigma_1 \in G} \sigma \sigma_1 \eta (\sigma \sigma_1)^{-1} \sigma = \tau \sigma$
- ii) $\tau v_1 = 0, v_1 \in V_1$. For $\sigma^{-1} v_1 \in V_1, \sigma \in G$, so that $\eta \sigma^{-1} v_1 = 0 \Rightarrow \tau v_1 = 0$
- iii) $(1-\tau)v \in V_1, v \in V, 1$ denoting the identity of G . For $(1-\eta)v \in V_1$, so that $(1-\eta)\sigma^{-1}v \in V_1 \Rightarrow \sigma(1-\eta)\sigma^{-1}v \in V_1, \sigma \in G$. It follows that $(1-\tau)v = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(1-\eta)\sigma^{-1}v \in V_1$.

Let $V'_2 = \tau V$. V'_2 is invariant under G as $\sigma(\tau v) = \tau(\sigma v)$. For any $v, v = \tau v + (1-\tau)v$. It follows from iii) that $V = V_1 + V'_2$. ii), iii) imply $\tau(1-\tau) = 0 \Leftrightarrow \tau = \tau^2$. Hence $\tau v'_2 = v'_2$ for $v'_2 \in V'_2$. Let $v_1 + v'_2 = 0$, where $v_1 \in V_1, v'_2 \in V'_2$. Applying τ to both sides, we get $v'_2 = 0$ and so $v_1 = 0$. Hence $V = V_1 \oplus V'_2$.

Repeated application of Maschke's Theorem yields the

COROLLARY. *Let G be a finite group acting on the finite-dimensional vector space V . Then $V = V_1 \oplus \dots \oplus V_s$, the V_i 's being invariant subspaces of V and G acting irreducibly on each V_i .*

For finite reflection groups, we have

THEOREM 2.7. *Let G be a finite reflection group acting on V . There exists a decomposition $V = V_1 \oplus \dots \oplus V_s$ into invariant subspaces such that:*

- 1) *Let $G_i = G|_{V_i}$ = group of restrictions of elements of G to V_i . Then G is isomorphic to $G_1 \times \dots \times G_s$*
- 2) *Each $G_i, 1 \leq i \leq s$, is a reflection group acting irreducibly on V_i .*

Proof. By the corollary to Theorem 2.6, there exists a decomposition $V = V_1 \oplus \dots \oplus V_s$, the V_i 's being invariant subspaces and G_i irreducible for $1 \leq i \leq s$. We label the V_i 's so that V_1, \dots, V_r are 1-dimensional and $G|_{V_i} = \text{identity}$.

By the remark following Definition 2.1, for each reflection σ there exists an eigenvector $v \in V - \pi, \pi$ being the r.h. for σ . Call v a root of G . We have

$$(2.22) \quad \dim(V_i + \pi) + \dim(V_i \cap \pi) = \dim V_i + \dim \pi.$$

If $V_i \not\subset \pi$, then $V_i + \pi = V$ and we conclude from (2.22) that $\dim V_i = \dim (V_i \cap \pi) + 1$. I.e. $V_i \cap \pi$ is a hyperplane in V_i and $\sigma|_{V_i}$ a reflection on V_i . Choose $u \in V_i - \pi$ so that u is an eigenvector of σ . u is a multiple of the root v , so that $v \in V_i$. Thus $\sigma|_{V_i}$ is a reflection of V_i if $v \in V_i$, and the identity if $v \notin V_i$. Furthermore, each root v is in some V_i , $r + 1 \leq i \leq s$, otherwise the corresponding reflection σ would have been the identity.

Let $\tilde{G}_i =$ subgroup generated by those reflections whose roots are in V_i , $1 \leq i \leq s$. It is readily checked that $G = \tilde{G}_1 \times \dots \times \tilde{G}_s$, $G_i = \tilde{G}_i|_{V_i}$. If $\sigma \in \tilde{G}_i$ and $\sigma|_{V_i} =$ identity then $\sigma =$ identity. The mapping $\sigma \rightarrow \sigma|_{V_i}$ is thus an isomorphism from \tilde{G}_i onto G_i .

THEOREM 2.8. *Let G be a finite reflection group acting on V and decompose V as in Theorem 2.7. Every polynomial invariant under G is a polynomial in the invariant polynomials of G_1, \dots, G_s .*

Proof. For each $v \in V$, write $v = v_1 + \dots + v_s$, $v_i \in V_i$. By Theorem 2.7, for each $\sigma \in G$, we may write $\sigma v = \sigma_1 v_1 + \dots + \sigma_s v_s$, $\sigma_i \in G_i$. For any polynomial function $p(v)$ on V , we have $p(v) = \sum_{i=1}^N p_{i1}(v_1) \dots p_{is}(v_s)$ where $p_{ij}(v_j)$ is a polynomial function on V_j . If $p(v)$ is invariant under G , then

$$(2.23) \quad p(v) = \frac{1}{|G|} \sum_{\sigma \in G} p(\sigma v) = \sum_{i=1}^N I_{i1}(v_1) \dots I_{is}(v_s)$$

where

$$(2.24) \quad I_{ij}(v_j) = \frac{1}{|G_j|} \sum_{\sigma_j \in G_j} p_{ij}(\sigma_j v_j)$$

is an invariant of G_j .

CHAPTER III

THE DEGREES OF THE BASIC INVARIANTS

We determine the degrees of the basic homogeneous invariants in case G is a finite reflection group. We present two different methods. The first one (Theorem 3.8), restricts itself to the case where k is the real field and has the advantage of providing an effective method for computing the