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Let  $J_i = \sum A_m y_n^m$ , the  $A_m$ 's being polynomials in  $y_1, \dots, y_{n-1}$ . (2.20) implies that  $A_m = 0$  whenever  $h \nmid m$ , so that  $A_m = 0$ ,  $0 \leq m \leq h-1$ . Since

$$\frac{\partial J_i}{\partial y_m} = \sum_m A_m y_n^{m-1},$$

we conclude

$$y_n^{h-1} \left| \frac{\partial J_i}{\partial y_n}, 1 \leq i \leq n \right.$$

Hence

$$(2.21) \quad y_n^{h-1} \left| \frac{\partial (J_1, \dots, J_n)}{\partial (y_1, \dots, y_n)}, \right.$$

Since

$$\frac{\partial (J_1, \dots, J_n)}{\partial (y_1, \dots, y_n)} = J(x) \cdot \det \tau,$$

(2.21) is equivalent to  $L^{h-1}(x) \mid J(x)$ . It follows that if  $L_i(x) = 0$ ,  $1 \leq i \leq r$ , are the r.h.'s for  $G$ , then  $\prod_{i=1}^r L_i \mid J$ . But  $J$ ,  $\prod_{i=1}^r L_i$  have the same degree  $r$ , so that  $J = c \prod_{i=1}^r L_i c \neq 0$ .

#### 4. DECOMPOSITION OF FINITE REFLECTION GROUPS

We shall decompose every finite reflection group into a direct product of irreducible ones and show that it suffices to study the invariant theory of the irreducible groups.

**DEFINITION 2.3.** Let the group  $G$  act on  $V$ .  $G$  is said to be reducible iff there exists a proper subspace  $W$  invariant under  $G$ ; i.e.  $\sigma w \in W$  for  $\sigma \in G$ ,  $w \in W$ .  $G$  is said to be completely reducible iff  $V = V_1 \oplus V_2$ ,  $V_1$  and  $V_2$  being proper invariant subspaces.  $G$  is said to be irreducible iff it is not reducible.

**THEOREM 2.6.** (Maschke [22], Vol. 2, p. 179). *Let  $G$  be a finite group acting on the vector space  $V$ . If  $G$  is reducible, then it is completely reducible.*

*Proof.* Let  $V_1$  be a proper invariant subspace of  $V$ . Let  $V_2$  be a complementary subspace. Thus for  $v \in V$ , we have a unique decomposition

$v = v_1 + v_2, v_i \in V_i (i=1, 2)$ . Let  $\eta v = v_2$  and set  $\tau = \frac{1}{|G|} \sum_{\sigma \in G} \sigma \eta \sigma^{-1}$ .

$\tau$  satisfies the following:

- i)  $\tau \sigma = \sigma \tau, \sigma \in G$ . For  $\sigma \tau = \frac{1}{|G|} \sum_{\sigma_1 \in G} \sigma \sigma_1 \eta (\sigma \sigma_1)^{-1} \sigma = \tau \sigma$
- ii)  $\tau v_1 = 0, v_1 \in V_1$ . For  $\sigma^{-1} v_1 \in V_1, \sigma \in G$ , so that  $\eta \sigma^{-1} v_1 = 0 \Rightarrow \tau v_1 = 0$
- iii)  $(1 - \tau) v \in V_1, v \in V$ , 1 denoting the identity of  $G$ . For  $(1 - \eta) v \in V_1$ , so that  $(1 - \eta) \sigma^{-1} v \in V_1 \Rightarrow \sigma (1 - \eta) \sigma^{-1} v \in V_1, \sigma \in G$ . It follows that  $(1 - \tau) v = \frac{1}{|G|} \sum_{\sigma \in G} \sigma (1 - \eta) \sigma^{-1} v \in V_1$ .

Let  $V'_2 = \tau V$ .  $V'_2$  is invariant under  $G$  as  $\sigma(\tau v) = \tau(\sigma v)$ . For any  $v$ ,  $v = \tau v + (1 - \tau) v$ . It follows from iii) that  $V = V_1 + V'_2$ . ii), iii) imply  $\tau(1 - \tau) = 0 \Leftrightarrow \tau = \tau^2$ . Hence  $\tau v'_2 = v'_2$  for  $v'_2 \in V'_2$ . Let  $v_1 + v'_2 = 0$ , where  $v_1 \in V_1, v'_2 \in V'_2$ . Applying  $\tau$  to both sides, we get  $v'_2 = 0$  and so  $v_1 = 0$ . Hence  $V = V_1 \oplus V'_2$ .

Repeated application of Maschke's Theorem yields the

**COROLLARY.** *Let  $G$  be a finite group acting on the finite-dimensional vector space  $V$ . Then  $V = V_1 \oplus \dots \oplus V_s$ , the  $V_i$ 's being invariant subspaces of  $V$  and  $G$  acting irreducibly on each  $V_i$ .*

For finite reflection groups, we have

**THEOREM 2.7.** *Let  $G$  be a finite reflection group acting on  $V$ . There exists a decomposition  $V = V_1 \oplus \dots \oplus V_s$  into invariant subspaces such that :*

- 1) *Let  $G_i = G|_{V_i}$  = group of restrictions of elements of  $G$  to  $V_i$ . Then  $G$  is isomorphic to  $G_1 \times \dots \times G_s$*
- 2) *Each  $G_i, 1 \leq i \leq s$ , is a reflection group acting irreducibly on  $V_i$ .*

*Proof.* By the corollary to Theorem 2.6, there exists a decomposition  $V = V_1 \oplus \dots \oplus V_s$ , the  $V_i$ 's being invariant subspaces and  $G_i$  irreducible for  $1 \leq i \leq s$ . We label the  $V_i$ 's so that  $V_1, \dots, V_r$  are 1-dimensional and  $G|_{V_i}$  = identity.

By the remark following Definition 2.1, for each reflection  $\sigma$  there exists an eigenvector  $v \in V - \pi$ ,  $\pi$  being the r.h. for  $\sigma$ . Call  $v$  a root of  $G$ . We have

$$(2.22) \quad \dim(V_i + \pi) + \dim(V_i \cap \pi) = \dim V_i + \dim \pi.$$

If  $V_i \notin \pi$ , then  $V_i + \pi = V$  and we conclude from (2.22) that  $\dim V_i = \dim (V_i \cap \pi) + 1$ . I.e.  $V_i \cap \pi$  is a hyperplane in  $V_i$  and  $\sigma|_{V_i}$  a reflection on  $V_i$ . Choose  $u \in V_i - \pi$  so that  $u$  is an eigenvector of  $\sigma$ .  $u$  is a multiple of the root  $v$ , so that  $v \in V_i$ . Thus  $\sigma|_{V_i}$  is a reflection of  $V_i$  if  $v \in V_i$ , and the identity if  $v \notin V_i$ . Furthermore, each root  $v$  is in some  $V_i$ ,  $r+1 \leq i \leq s$ , otherwise the corresponding reflection  $\sigma$  would have been the identity.

Let  $\tilde{G}_i =$  subgroup generated by those reflections whose roots are in  $V_i$ ,  $1 \leq i \leq s$ . It is readily checked that  $G = \tilde{G}_1 \times \dots \times \tilde{G}_s$ ,  $G_i = \tilde{G}_i|_{V_i}$ . If  $\sigma \in \tilde{G}_i$  and  $\sigma|_{V_i} =$  identity then  $\sigma =$  identity. The mapping  $\sigma \rightarrow \sigma|_{V_i}$  is thus an isomorphism from  $\tilde{G}_i$  onto  $G_i$ .

**THEOREM 2.8.** *Let  $G$  be a finite reflection group acting on  $V$  and decompose  $V$  as in Theorem 2.7. Every polynomial invariant under  $G$  is a polynomial in the invariant polynomials of  $G_1, \dots, G_s$ .*

*Proof.* For each  $v \in V$ , write  $v = v_1 + \dots + v_s$ ,  $v_i \in V_i$ . By Theorem 2.7, for each  $\sigma \in G$ , we may write  $\sigma v = \sigma_1 v_1 + \dots + \sigma_s v_s$ ,  $\sigma_i \in G_i$ . For any polynomial function  $p(v)$  on  $V$ , we have  $p(v) = \sum_{i=1}^N p_{i1}(v_1) \dots p_{is}(v_s)$  where  $p_{ij}(v_j)$  is a polynomial function on  $V_j$ . If  $p(v)$  is invariant under  $G$ , then

$$(2.23) \quad p(v) = \frac{1}{|G|} \sum_{\sigma \in G} p(\sigma v) = \sum_{i=1}^N I_{i1}(v_1) \dots I_{is}(v_s)$$

where

$$(2.24) \quad I_{ij}(v_j) = \frac{1}{|G_j|} \sum_{\sigma_j \in G_j} p_{ij}(\sigma_j v_j)$$

is an invariant of  $G_j$ .

## CHAPTER III

### THE DEGREES OF THE BASIC INVARIANTS

We determine the degrees of the basic homogeneous invariants in case  $G$  is a finite reflection group. We present two different methods. The first one (Theorem 3.8), restricts itself to the case where  $k$  is the real field and has the advantage of providing an effective method for computing the