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Let $J_i = \sum A_m y_n^m$, the A_m's being polynomials in $y_1, ..., y_{n-1}$. (2.20) implies that $A_m = 0$ whenever $h \nmid m$, so that $A_m = 0, 0 \leqslant m \leqslant h - 1$. Since

$$\frac{\partial J_i}{\partial y_m} = \Sigma_m A_m y_n^{m-1},$$

we conclude

$$y_n^{h-1} \Bigg| \frac{\partial J_i}{\partial y_n}, \ 1 \leqslant i \leqslant n .$$

Hence

(2.21)
$$y_n^{h-1} \Bigg| \frac{\partial (J_1, ..., J_n)}{\partial (y_1, ..., y_n)},$$

Since

$$\frac{\partial (J_1, ..., J_n)}{\partial (y_1, ..., y_n)} = J(x) \cdot \det \tau ,$$

(2.21) is equivalent to $L^{h-1}(x) \mid J(x)$. It follows that if $L_i(x) = 0$, $1 \leqslant i \leqslant r$, are the r.h.'s for G, then $\prod_{i=1}^{r} L_i \mid J$. But J, $\prod_{i=1}^{r} L_i$ have the same degree r, so that $J = c \prod_{i=1}^{r} L_i \; c \neq 0$.

4. Decomposition of Finite Reflection Groups

We shall decompose every finite reflection group into a direct product of irreducible ones and show that it suffices to study the invariant theory of the irreducible groups.

Definition 2.3. Let the group G act on V. G is said to be reducible iff there exists a proper subspace W invariant under G; i.e. $\sigma w \in W$ for $\sigma \in G$, $w \in W$. G is said to be completely reducible iff $V = V_1 \oplus V_2$, V_1 and V_2 being proper invariant subspaces. G is said to be irreducible iff it is not reducible.

Theorem 2.6. (Maschke [22], Vol. 2, p. 179). *Let G be a finite group acting on the vector space V. If G is reducible, then it is completely reducible.*

Proof. Let V_1 be a proper invariant subspace of V. Let V_2 be a complementary subspace. Thus for $v \in V$, we have a unique decomposition

$v = v_1 + v_2, v_i \in V_i\,(i=1,2)$. Let $\eta\, v = v_2$ and set $\tau = \frac{1}{|G|} \sum_{\sigma \varepsilon G} \sigma\, \eta\, \sigma^{-1}$.

τ satisfies the following:

i) $\tau\,\sigma = \sigma\,\tau, \sigma \in G$. For $\sigma\,\tau = \frac{1}{|G|} \sum_{\sigma_1 \varepsilon G} \sigma\,\sigma_1\,\eta\,(\sigma\sigma_1)^{-1}\,\sigma = \tau\,\sigma$

ii) $\tau\, v_1 = 0, v_1 \in V_1$. For $\sigma^{-1} v_1 \in V_1, \sigma \in G$, so that $\eta\,\sigma^{-1} v_1 = 0$ $\Rightarrow \tau\, v_1 = 0$

iii) $(1-\tau)\, v \in V_1, v \in V$, 1 denoting the identity of G. For $(1-\eta)\, v \in V_1$, so that $(1-\eta)\,\sigma^{-1}\, v \in V_1 \Rightarrow \sigma\,(1-\eta)\,\sigma^{-1}\, v \in V_1, \sigma \in G$. It follows that $(1-\tau)\, v = \frac{1}{|G|} \sum_{\sigma \varepsilon G} \sigma\,(1-\eta)\sigma^{-1}\, v \in V_1$.

Let $V_2' = \tau\, V$. V_2' is invariant under G as $\sigma\,(\tau v) = \tau\,(\sigma v)$. For any v, $v = \tau\, v + (1-\tau)\, v$. It follows from iii) that $V = V_1 + V_2'$. ii), iii) imply $\tau\,(1-\tau) = 0 \Leftrightarrow \tau = \tau^2$. Hence $\tau\, v_2' = v_2'$ for $v_2' \in V_2'$. Let $v_1 + v_2' = 0$, where $v_1 \in V_1, v_2' \in V_2'$. Applying τ to both sides, we get $v_2' = 0$ and so $v_1 = 0$. Hence $V = V_1 \oplus V_2'$.

Repeated application of Maschke's Theorem yields the

Corollary. *Let G be a finite group acting on the finite-dimensional vector space V. Then $V = V_1 \oplus \ldots \oplus V_s$, the V_i's being invariant subspaces of V and G acting irreducibly on each V_i.*

For finite reflection groups, we have

Theorem 2.7. *Let G be a finite reflection group acting on V. There exists a decomposition $V = V_1 \oplus \ldots \oplus V_s$ into invariant subspaces such that:*

1) *Let $G_i = G\,|_{V_i}$ = group of restrictions of elements of G to V_i. Then G is isomorphic to $G_1 \times \ldots \times G_s$*

2) *Each G_i, $1 \leqslant i \leqslant s$, is a reflection group acting irreducibly on V_i.*

Proof. By the corollary to Theorem 2.6, there exists a decomposition $V = V_1 \oplus \ldots \oplus V_s$, the V_i's being invariant subspaces and G_i irreducible for $1 \leqslant i \leqslant s$. We label the V_i's so that $V_1, \ldots, V_r$ are 1-dimensional and $G\,|_{V_i}$ = identity.

By the remark following Definition 2.1, for each reflection σ there exists an eigenvector $v \in V - \pi$, π being the r.h. for σ. Call v a root of G. We have

$$\dim\,(V_i + \pi) + \dim\,(V_i \cap \pi) = \dim V_i + \dim \pi\,. \tag{2.22}$$

If $V_i \not\subset \pi$, then $V_i + \pi = V$ and we conclude from (2.22) that $\dim V_i = \dim (V_i \cap \pi) + 1$. I.e. $V_i \cap \pi$ is a hyperplane in V_i and $\sigma |_{V_i}$ a reflection on V_i. Choose $u \in V_i - \pi$ so that u is an eigenvector of σ. u is a multiple of the root v, so that $v \in V_i$. Thus $\sigma |_{V_i}$ is a reflection of V_i if $v \in V_i$, and the identity if $v \notin V_i$. Furthermore, each root v is in some V_i, $r + 1 \leqslant i \leqslant s$, otherwise the corresponding reflection σ would have been the identity.

Let $\tilde{G}_i$ = subgroup generated by those reflections whose roots are in V_i, $1 \leqslant i \leqslant s$. It is readily checked that $G = \tilde{G}_1 \times \ldots \times \tilde{G}_s$, $G_i = \tilde{G}_i |_{V_i}$. If $\sigma \in \tilde{G}_i$ and $\sigma |_{V_i}$ = identity then σ = identity. The mapping $\sigma \to \sigma |_{V_i}$ is thus an isomorphism from $\tilde{G}_i$ onto G_i.

THEOREM 2.8. *Let G be a finite reflection group acting on V and decompose V as in Theorem 2.7. Every polynomial invariant under G is a polynomial in the invariant polynomials of $G_1, \ldots, G_s$.*

Proof. For each $v \in V$, write $v = v_1 + \ldots + v_s$, $v_i \in V_i$. By Theorem 2.7, for each $\sigma \in G$, we may write $\sigma v = \sigma_1 v_1 + \ldots + \sigma_s v_s$, $\sigma_i \in G_i$. For any polynomial function $p(v)$ on V, we have $p(v) = \sum_{i=1}^{N} p_{i1}(v_1) \ldots p_{is}(v_s)$ where $p_{ij}(v_j)$ is a polynomial function on V_j. If $p(v)$ is invariant under G, then

$$(2.23) \qquad p(v) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} p(\sigma v) = \sum_{i=1}^{N} I_{i1}(v_1) \ldots I_{is}(v_s)$$

where

$$(2.24) \qquad I_{ij}(v_j) = \frac{1}{|G_j|} \sum_{\sigma_j \varepsilon G_j} p_{ij}(\sigma_j v_j)$$

is an invariant of G_j.

CHAPTER III

THE DEGREES OF THE BASIC INVARIANTS

We determine the degrees of the basic homogeneous invariants in case G is a finite reflection group. We present two different methods. The first one (Theorem 3.8), restricts itself to the case where k is the real field and has the advantage of providing an effective method for computing the