

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 24 (1978)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INVARIANTS OF FINITE REFLECTION GROUPS
Autor: Flatto, Leopold
Kapitel: 3. A Formula for $\frac{\delta \left(l_1, \dots, l_n \right)}{\delta \left(x_1, \dots, x_n \right)}$
DOI: <https://doi.org/10.5169/seals-49704>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 12.03.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

3. A FORMULA FOR $\frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)}$

We obtain a formula which shall be used in Chapter III.

THEOREM 2.5. Let G be a finite reflection group acting on the n -dimensional space V . Let I_1, \dots, I_n be a basic set of homogeneous invariants for G . Let x be a coordinate system for V and $L_i(x) = 0, 1 \leq i \leq r$, the r.h.'s for G , each L_i being linear and homogeneous. Then

$$(2.19) \quad \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)} = c \prod_{i=1}^r L_i(x)$$

c being a constant $\neq 0$.

Proof. Let J the left hand side of (2.19). We observe that J is a non-zero homogeneous polynomial of degree $\sum_{i=1}^n (d_i - 1)$. By Theorem 2.2,

$\sum_{i=1}^n (d_i - 1) = r$, so that $\deg J = r$. If k is the real field R , we have the following simple proof of (2.19). $I_i = I_i(x_1, \dots, x_n), 1 \leq i \leq n$, is a mapping from x -space to I -space. This mapping is not 1 - 1 in any neighborhood of a point x lying in the r.h. $L_i(x) = 0$, as any point and its reflection get mapped into the same point I . It follows from the Implicit Function Theorem that $J(x) = 0$ whenever $L_i(x) = 0$. Thus $L_i \mid J, 1 \leq i \leq r$,

and so $\prod_{i=1}^r L_i \mid J$. Since $J, \prod_{i=1}^r L_i$ have the same degree r , we have

$$J = c \prod_{i=1}^r L_i, c \neq 0.$$

For an arbitrary field k , the theorem is proven as follows. Let π be an r.h. with equation $L(x) = 0$ and H the subgroup of h elements in G fixing π . Thus there are $h - 1$ reflections in G with r.h. π . We show that $L^{h-1} \mid J$. By Lemma 2.2, H is a cyclic group generated by an element σ . Furthermore there exists $v \notin \pi$ and a primitive h -th root of 1 such that $\sigma(v) = \zeta v$. Choose a coordinate system $y = (y_1, \dots, y_n)$ in V so that π has the equation $y_n = 0$ and $v = (0, \dots, 0, 1)$ σ then becomes the transformation $(y_1, \dots, y_{n-1}, y_n) \rightarrow (y_1, \dots, y_{n-1}, \zeta y_n)$. Let $x = \tau y$ and $J_i(y) = I_i(\tau y), 1 \leq i \leq n$. We have

$$(2.20) \quad J_i(y_1, \dots, y_{n-1}, \zeta y_n) = J_i(y_1, \dots, y_{n-1}, y_n), 1 \leq i \leq n$$

Let $J_i = \sum A_m y_n^m$, the A_m 's being polynomials in y_1, \dots, y_{n-1} . (2.20) implies that $A_m = 0$ whenever $h \nmid m$, so that $A_m = 0, 0 \leq m \leq h - 1$. Since

$$\frac{\partial J_i}{\partial y_m} = \sum_m A_m y_n^{m-1},$$

we conclude

$$y_n^{h-1} \left| \frac{\partial J_i}{\partial y_n}, 1 \leq i \leq n. \right.$$

Hence

$$(2.21) \quad y_n^{h-1} \left| \frac{\partial (J_1, \dots, J_n)}{\partial (y_1, \dots, y_n)}, \right.$$

Since

$$\frac{\partial (J_1, \dots, J_n)}{\partial (y_1, \dots, y_n)} = J(x) \cdot \det \tau,$$

(2.21) is equivalent to $L^{h-1}(x) \mid J(x)$. It follows that if $L_i(x) = 0, 1 \leq i \leq r$, are the r.h.'s for G , then $\prod_{i=1}^r L_i \mid J$. But $J, \prod_{i=1}^r L_i$ have the same degree r , so that $J = c \prod_{i=1}^r L_i, c \neq 0$.

4. DECOMPOSITION OF FINITE REFLECTION GROUPS

We shall decompose every finite reflection group into a direct product of irreducible ones and show that it suffices to study the invariant theory of the irreducible groups.

DEFINITION 2.3. Let the group G act on V . G is said to be reducible iff there exists a proper subspace W invariant under G ; i.e. $\sigma w \in W$ for $\sigma \in G, w \in W$. G is said to be completely reducible iff $V = V_1 \oplus V_2, V_1$ and V_2 being proper invariant subspaces. G is said to be irreducible iff it is not reducible.

THEOREM 2.6. (Maschke [22], Vol. 2, p. 179). *Let G be a finite group acting on the vector space V . If G is reducible, then it is completely reducible.*

Proof. Let V_1 be a proper invariant subspace of V . Let V_2 be a complementary subspace. Thus for $v \in V$, we have a unique decomposition