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3. A FORMULA FOR $\frac{\partial(I_1, \dots, I_n)}{\partial(x_1, \dots, x_n)}$

We obtain a formula which shall be used in Chapter III.

THEOREM 2.5. Let G be a finite reflection group acting on the n -dimensional space V . Let I_1, \dots, I_n be a basic set of homogeneous invariants for G . Let x be a coordinate system for V and $L_i(x) = 0$, $1 \leq i \leq r$, the r.h.'s for G , each L_i being linear and homogeneous. Then

$$(2.19) \quad \frac{\partial(I_1, \dots, I_n)}{\partial(x_1, \dots, x_n)} = c \prod_{i=1}^r L_i(x)$$

c being a constant $\neq 0$.

Proof. Let J the left hand side of (2.19). We observe that J is a non-zero homogeneous polynomial of degree $\sum_{i=1}^n (d_i - 1)$. By Theorem 2.2, $\sum_{i=1}^n (d_i - 1) = r$, so that $\deg J = r$. If k is the real field R , we have the following simple proof of (2.19). $I_i = I_i(x_1, \dots, x_n)$, $1 \leq i \leq n$, is a mapping from x -space to I -space. This mapping is not $1 - 1$ in any neighborhood of a point x lying in the r.h. $L_i(x) = 0$, as any point and its reflection get mapped into the same point I . It follows from the Implicit Function Theorem that $J(x) = 0$ whenever $L_i(x) = 0$. Thus $L_i | J$, $1 \leq i \leq r$, and so $\prod_{i=1}^r L_i | J$. Since J , $\prod_{i=1}^r L_i$ have the same degree r , we have $J = c \prod_{i=1}^r L_i$, $c \neq 0$.

For an arbitrary field k , the theorem is proven as follows. Let π be an r.h. with equation $L(x) = 0$ and H the subgroup of h elements in G fixing π . Thus there are $h - 1$ reflections in G with r.h. π . We show that $L^{h-1} | J$. By Lemma 2.2, H is a cyclic group generated by an element σ . Furthermore there exists $v \notin \pi$ and a primitive h -th root of 1 such that $\sigma(v) = \zeta v$. Choose a coordinate system $y = (y_1, \dots, y_n)$ in V so that π has the equation $y_n = 0$ and $v = (0, \dots, 0, 1)$. σ then becomes the transformation $(y_1, \dots, y_{n-1}, y_n) \rightarrow (y_1, \dots, y_{n-1}, \zeta y_n)$. Let $x = \tau y$ and $J_i(y) = I_i(\tau y)$, $1 \leq i \leq n$. We have

$$(2.20) \quad J_i(y_1, \dots, y_{n-1}, \zeta y_n) = J_i(y_1, \dots, y_{n-1}, y_n), \quad 1 \leq i \leq n$$

Let $J_i = \sum A_m y_n^m$, the A_m 's being polynomials in y_1, \dots, y_{n-1} . (2.20) implies that $A_m = 0$ whenever $h \nmid m$, so that $A_m = 0$, $0 \leq m \leq h-1$. Since

$$\frac{\partial J_i}{\partial y_m} = \sum_m A_m y_n^{m-1},$$

we conclude

$$y_n^{h-1} \left| \frac{\partial J_i}{\partial y_n}, 1 \leq i \leq n \right.$$

Hence

$$(2.21) \quad y_n^{h-1} \left| \frac{\partial (J_1, \dots, J_n)}{\partial (y_1, \dots, y_n)}, \right.$$

Since

$$\frac{\partial (J_1, \dots, J_n)}{\partial (y_1, \dots, y_n)} = J(x) \cdot \det \tau,$$

(2.21) is equivalent to $L^{h-1}(x) | J(x)$. It follows that if $L_i(x) = 0$, $1 \leq i \leq r$, are the r.h.'s for G , then $\prod_{i=1}^r L_i | J$. But J , $\prod_{i=1}^r L_i$ have the same degree r , so that $J = c \prod_{i=1}^r L_i c \neq 0$.

4. DECOMPOSITION OF FINITE REFLECTION GROUPS

We shall decompose every finite reflection group into a direct product of irreducible ones and show that it suffices to study the invariant theory of the irreducible groups.

DEFINITION 2.3. Let the group G act on V . G is said to be reducible iff there exists a proper subspace W invariant under G ; i.e. $\sigma w \in W$ for $\sigma \in G$, $w \in W$. G is said to be completely reducible iff $V = V_1 \oplus V_2$, V_1 and V_2 being proper invariant subspaces. G is said to be irreducible iff it is not reducible.

THEOREM 2.6. (Maschke [22], Vol. 2, p. 179). *Let G be a finite group acting on the vector space V . If G is reducible, then it is completely reducible.*

Proof. Let V_1 be a proper invariant subspace of V . Let V_2 be a complementary subspace. Thus for $v \in V$, we have a unique decomposition