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3. A Formula for $\dfrac{\partial(I_1, \ldots, I_n)}{\partial(x_1, \ldots, x_n)}$

We obtain a formula which shall be used in Chapter III.

THEOREM 2.5. *Let* G *be a finite reflection group acting on the* n*-dimensional space* V. *Let* $I_1, \ldots, I_n$ *be a basic set of homogeneous invariants for* G. *Let* x *be a coordinate system for* V *and* $L_i(x) = 0$, $1 \leqslant i \leqslant r$, *the r.h.'s for* G, *each* L_i *being linear and homogeneous. Then*

$$\frac{\partial(I_1, \ldots, I_n)}{\partial(x_1, \ldots, x_n)} = c \prod_{i=1}^{r} L_i(x) \tag{2.19}$$

c being a constant $\neq 0$.

Proof. Let J the left hand side of (2.19). We observe that J is a non-zero homogeneous polynomial of degree $\sum_{i=1}^{n} (d_i - 1)$. By Theorem 2.2, $\sum_{i=1}^{n} (d_i - 1) = r$, so that $\deg J = r$. If k is the real field R, we have the following simple proof of (2.19). $I_i = I_i(x_1, \ldots, x_n)$, $1 \leqslant i \leqslant n$, is a mapping from x-space to I-space. This mapping is not $1 - 1$ in any neighborhood of a point x lying in the r.h. $L_i(x) = 0$, as any point and its reflection get mapped into the same point I. It follows from the Implicit Function Theorem that $J(x) = 0$. whenever $L_i(x) = 0$. Thus $L_i \mid J$, $1 \leqslant i \leqslant r$, and so $\prod_{i=1}^{r} L_i \mid J$. Since J, $\prod_{i=1}^{r} L_i$ have the same degree r, we have $J = c \prod_{i=1}^{r} L_i$, $c \neq 0$.

For an arbitrary field k, the theorem is proven as follows. Let π be an r.h. with equation $L(x) = 0$ and H the subgroup of h elements in G fixing π. Thus there are $h - 1$ reflections in G with r.h. π. We show that $L^{h-1} \mid J$. By Lemma 2.2, H is a cyclic group generated by an element σ. Furthermore there exists $v \notin \pi$ and a primitive h-th root of 1 such that $\sigma(v) = \zeta v$. Choose a coordinate system $y = (y_1, \ldots, y_n)$ in V so that π has the equation $y_n = 0$ and $v = (0, \ldots, 0, 1)$ σ then becomes the transformation $(y_1, \ldots, y_{n-1}, y_n) \to (y_1, \ldots, y_{n-1}, \zeta y_n)$. Let $x = \tau y$ and $J_i(y) = I_i(\tau y)$, $1 \leqslant i \leqslant n$. We have

$$J_i(y_1, \ldots, y_{n-1}, \zeta y_n) = J_i(y_1, \ldots, y_{n-1}, y_n), \; 1 \leqslant i \leqslant n \tag{2.20}$$

Let $J_i = \sum A_m y_n^m$, the A_m's being polynomials in $y_1, ..., y_{n-1}$. (2.20) implies that $A_m = 0$ whenever $h \nmid m$, so that $A_m = 0, 0 \leqslant m \leqslant h - 1$. Since

$$\frac{\partial J_i}{\partial y_m} = \Sigma_m A_m y_n^{m-1},$$

we conclude

$$y_n^{h-1} \Bigg| \frac{\partial J_i}{\partial y_n}, \ 1 \leqslant i \leqslant n .$$

Hence

(2.21) $$y_n^{h-1} \Bigg| \frac{\partial (J_1, ..., J_n)}{\partial (y_1, ..., y_n)},$$

Since

$$\frac{\partial (J_1, ..., J_n)}{\partial (y_1, ..., y_n)} = J(x) \cdot \det \tau ,$$

(2.21) is equivalent to $L^{h-1}(x) \mid J(x)$. It follows that if $L_i(x) = 0$, $1 \leqslant i \leqslant r$, are the r.h.'s for G, then $\prod_{i=1}^{r} L_i \mid J$. But J, $\prod_{i=1}^{r} L_i$ have the same degree r, so that $J = c \prod_{i=1}^{r} L_i \ c \neq 0$.

4. Decomposition of Finite Reflection Groups

We shall decompose every finite reflection group into a direct product of irreducible ones and show that it suffices to study the invariant theory of the irreducible groups.

Definition 2.3. Let the group G act on V. G is said to be reducible iff there exists a proper subspace W invariant under G; i.e. $\sigma w \in W$ for $\sigma \in G$, $w \in W$. G is said to be completely reducible iff $V = V_1 \oplus V_2$, V_1 and V_2 being proper invariant subspaces. G is said to be irreducible iff it is not reducible.

Theorem 2.6. (Maschke [22], Vol. 2, p. 179). *Let G be a finite group acting on the vector space V. If G is reducible, then it is completely reducible.*

Proof. Let V_1 be a proper invariant subspace of V. Let V_2 be a complementary subspace. Thus for $v \in V$, we have a unique decomposition